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# On the Convergence of the Numerical Solution for a Certain Partial Differential Equation of Third Order

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# ON THE CONVERGENCE OF THE NUMERICAL SOLUTION FOR A CERTAIN PARTIAL DIFFERENTIAL EQUATION OF THIRD ORDER

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## Introduction

The equation we are dealing with is relevant to quasigeostrophic motion of the atmosphere. It is

$$\frac{d}{dt}(\triangle\psi - k^2\psi) = 0 . \tag{1}$$

The mathematical formulation of the atmospheric motion and derivation of the above equation using geostrophic approximations can be found in reference [1].

Equation (1) is called the geostrophic conservation equation where d/dt denotes the following operator:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \tag{1.a}$$

and

$$u = -\frac{\partial\psi}{\partial y} \; ; \; v = \frac{\partial\psi}{\partial x} \tag{1.b}$$

$\triangle$ is the Laplacian operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$, and $k^2$ is a constant determined physically from Coriolis parameter f, acceleration of gravity g, and mean height of the atmosphere $h_o$, i.e., $k^2 = f^2/gh_o$. We call $(\triangle - k^2)$ the Helmholtzian operator. Equations (1) and (1.b) show that $\psi(x,y;t)$ is a stream function and u and v represent velocity components in the x and y directions.

It has been shown in [2] and [3] that it is reasonable to solve the above equation when $\psi$ is specified initially over a rectangular domain G, i.e., $-1 \le x \le 1$, $-1 \le y \le 1$ and also on the boundary of G for all time, i.e., $0 \le t \le T$, and when $\triangle\psi$, which denotes vorticity, is specified on the boundary as a function of time, only when fluid is entering the region, but not when it is leaving the region.

Ch. B. Sensenig [2] proved an existence and uniqueness theorem for the above problem when the domain is a half plane,

e.g., $x \geq 0$. Hence we shall solve the problem for the square numerically, i.e., we shall show that a particular finite difference scheme provides an approximate solution of the above problem accurate within a reasonable error under the assumption of the existence of a sufficiently smooth solution.

The general procedure of this paper is divided into two parts: development of the error formula for Helmholtzian (Part I - chapters 1-5) and establishment of estimates for the error itself and its difference quotients (part II - chapters 6-12). One of the interesting features of the convergence proof, which is necessitated by the non-linearity of the differential equation (1), is that we have to estimate up to the boundary the bounds for the first difference quotients of the solution of the finite difference analogue of the Poisson equation, in terms of a bound for the right hand side.

We shall start the first part by introducing in chapter 1 a particular finite difference scheme. By assuming that the true solution of (1) has continuous bounded derivatives up to order 4 and that the 4th derivatives satisfy a Lipschitz condition, we compute the truncation error (denoted by $T(\psi)$) and find that it is of order h, i.e., $T(\psi) = O(h)$.

In chapter 2 we analyze step by step the growth of the error for the Helmholtzian assuming that initial error is zero and that the difference equation which approximates the differential equation is solved exactly. Under the restrictions that $\Delta t \leq \Delta x/[\max(|\Delta/\Delta x\Psi|+|\Delta/\Delta y\Psi|)]$ and $T < 1/N_0$ (where $\Psi$ denotes the solution of the difference equation, T -- the total time after n time steps, i.e., $T = n\Delta t$, and $N_0$ is a constant

depending on the bounds of the derivatives of the true solution) we find the following formula governing the error of the Helmholtzian, i.e., $|\Delta_h e^{(T)} - k^2 e^{(T)}| \leq K_o h^{1-N_oT}$, in which e designates the error $e = \psi - \Psi$.

In chapter 3 we consider the initial error of the Helmholtzian (denoted by $E^{(o)}$) to be present and by similar analysis as in chapter 2 we develop an error formula for the Helmholtzian and we shall conclude that if the initial error is of order O(h) and if the restrictions of the previous chapter are satisfied the error of the Helmholtzian is of the same order as that in chapter 2.

In chapter 4 by introducing a concept of "time layer" we show that it is possible to extend the total time (denoted by $\tilde{T}$) to $\tilde{T} \leq k/N_o$ (where k denotes the number of "time layers") and by a similar error analysis as before we find a verified error formula for the Helmholtzian.

Chapter 5 briefly shows that the above established convergence is subject to the requirement that the round-off be of order $O(h^4)$.

In chapter 6 we obtain from the bound for the Helmholtzian by means of maximum principle and auxiliary function a bound for the error e.

In chapters 7-10 we make the necessary preparations for establishing the estimates for the difference quotients of the error. We simplify and reduce the original problem to two seperate problems: the non-homogeneous and the homogeneous one. Solving the non-homogeneous problem we come across the solution of the so-called "discrete potential equation" whose properties

are discussed in chapter 2 using von der Pol's and A. Stöhr's results in [5] and [6], respectively.

In chapters 8 and 9 we obtain subsidiary bounds for the solution and its difference quotients of the non-homogeneous problem with the help of the bounds for the solution of the "discrete potential equation" given by A. Stöhr in [6].

In chapter 10 we obtain bounds for the difference quotients of the solution of the homogeneous problem applying the method given in Tamarkin-Feller [4].

Chapter 11 establishes the necessary estimates for the difference quotients of the error.

In chapter 12 we show that using the reflection principle we can extend the above mentioned estimates up to the boundary.

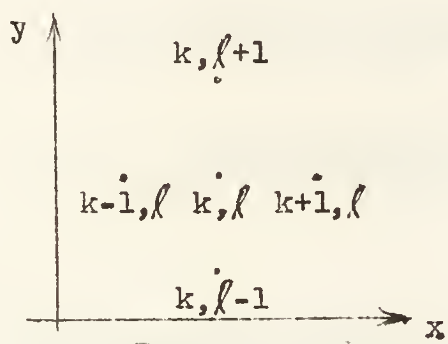
## PART I

### 1. Truncation Error

Let us define a rectangular grid of points by coordinates

$$(1.1)\qquad \begin{aligned} x_k &= k\Delta x \qquad & k &= 0,1,\ldots,p \\ y_\ell &= \ell\Delta y \qquad & \ell &= 0,1,\ldots,q \end{aligned}$$

and denote quantities at the point $(k,\ell)$ by subscrpt $k,\ell$. In the same manner we replace $t$ by $m\Delta t$, $m$ having integral values. Let us choose the mesh width to be $\Delta x = \Delta y = h$. From the following diagram




we approximate the derivatives in (1) using centered differences as follows:

$$(1.2)\qquad \begin{aligned} v = \psi_x &\cong \frac{\psi(x+\Delta x,y;t)-\psi(x-\Delta x,y;t)}{2\Delta x} = \frac{\Delta}{\Delta x}\psi = \frac{\psi_{k+1,\ell}-\psi_{k-1,\ell}}{2h} \\ -u = \psi_y &\cong \frac{\psi(x,y+\Delta y;t)-\psi(x,y-\Delta y;t)}{2\Delta y} = \frac{\Delta}{\Delta y}\psi = \frac{\psi_{k,\ell+1}-\psi_{k,\ell-1}}{2h} \end{aligned}$$

and we replace the Laplacian of quantity $G$ by the following finite difference approximation $\Delta_h$, where

$$(1.3)\qquad \Delta G \cong \Delta_h G = \frac{G_{k+1,\ell}+G_{k-1,\ell}+G_{k,\ell+1}+G_{k,\ell-1}-4G_{k\ell}}{h^2}$$

The differential operator $d/dt$ is approximated in the following manner taking into consideration the signs of $u$ and $v$.

<u>Case 1</u> If $u > 0$, $v > 0$ then

$$[\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}]g \cong \frac{g(x,y,t+\triangle t)-g(x,y,t)}{\triangle t}+$$

$$+ u(x,y,t)[\frac{g(x,y,t)-g(x-\triangle x,y,t)}{h}]+ \tag{1.4}$$

$$+ v(x,y,t)[\frac{g(x,y,t)-g(x,y-\triangle y,t)}{h}] \ .$$

<u>Case 2</u> If $u > 0$, $v < 0$ then

$$[\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}]g \cong \frac{g(x,y,t+\triangle t)-g(x,y,t)}{\triangle t}+$$

$$+ u(x,y,t)[\frac{g(x,y,t)-g(x-\triangle x,y,t)}{h}]+ \tag{1.5}$$

$$+ v(x,y,t)[\frac{g(x,y+\triangle y,t)-g(x,y,t)}{h}] \ .$$

<u>Case 3</u> If $u < 0$ and $v > 0$

$$[\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}]g \cong \frac{g(x,y,t+\triangle t)-g(x,y,t)}{\triangle t}+$$

$$+ u(x,y,t)[\frac{g(x+\triangle x,y,t)-g(x,y,t)}{h}]+ \tag{1.6}$$

$$+ v(x,y,t)[\frac{g(x,y,t)-g(x,y-\triangle y,t)}{h}] \ .$$

<u>Case 4</u> If $u < 0$ and $v < 0$

$$[\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}]g \cong g(x,y,t+\triangle t)-g(x,y,t)+$$

$$+ u(x,y,t)[\frac{g(x+\triangle x,y,t)-g(x,y,t)}{h}]+ \tag{1.7}$$

$$+ v(x,y,t)[\frac{g(x,y+\triangle y,t)-g(x,y,t)}{h}] \ .$$

Using the finite difference approximations (1.2) and (1.3) equation (1) becomes:

$$(1.8)\qquad \left[\frac{\triangle}{\triangle t}-\left(\frac{\triangle}{\triangle y}\psi\right)\frac{\triangle}{\triangle x}+\left(\frac{\triangle}{\triangle x}\psi\right)\frac{\triangle}{\triangle y}\right](\triangle_h\psi-k^2\psi) = T(\psi)$$

where $\triangle/\triangle x$ denotes the finite difference approximation to $\partial/\partial x$, and similarly

$$\frac{\partial}{\partial y} \cong \frac{\triangle}{\triangle y}\ ,\quad \frac{\partial}{\partial t} \cong \frac{\triangle}{\triangle t}$$

and $T(\psi)$ denotes the truncation error ($\psi$ satisfies the finite difference equation except for the truncation error). From (1.8) we shall calculate the truncation error. According to (1.3)

$$(1.9)\quad \triangle_h\psi-k^2\psi = \frac{1}{h^2}[\psi_{k+1,\ell}+\psi_{k-1,\ell}+\psi_{k,\ell+1}+\psi_{k,\ell-1}-4\psi_{k\ell}]-k^2\psi_{k\ell}$$

and if we assume that the function $\psi(x,y,t)$ has bounded derivatives up to order 4, we can apply Taylor's expansion and get:

$$\begin{aligned}\triangle_h\psi-k^2\psi = \frac{1}{h^2}\Big[&\psi_{k\ell}+\triangle x\frac{\partial\psi_{k\ell}}{\partial x}+\frac{(\triangle x)^2}{2}\frac{\partial^2\psi_{k\ell}}{\partial x^2}+\frac{(\triangle x)^3}{6}\frac{\partial^3\psi_{k\ell}}{\partial x^3}+\\ &+\frac{(\triangle x)^4}{24}\frac{\partial^4\bar{\psi}_{k,k+1,\ell}}{\partial x^4}+\psi_{k\ell}-\triangle x\frac{\partial\psi_{k\ell}}{\partial x}+\frac{(\triangle x)^2}{2}\frac{\partial^2\psi_{k\ell}}{\partial x^2}-\\ &-\frac{(\triangle x)^3}{6}\frac{\partial^3\psi_{k\ell}}{\partial x^3}+\frac{(\triangle x)^4}{24}\frac{\partial^4\bar{\psi}_{k,k-1,\ell}}{\partial x^4}+\psi_{k\ell}+\triangle y\frac{\partial\psi_{k\ell}}{\partial y}+\\ &+\frac{(\triangle y)^2}{2}\frac{\partial^2\psi_{k\ell}}{\partial x^2}+\frac{(\triangle y)^3}{6}\frac{\partial^3\psi_{k\ell}}{\partial y^3}+\frac{(\triangle y)^4}{24}\frac{\partial^4\bar{\psi}_{k,\ell+1,\ell}}{\partial y^4}+\\ &+\psi_{k\ell}-\triangle y\frac{\partial\psi_{k\ell}}{\partial y}+\frac{(\triangle y)^2}{2}\frac{\partial^2\psi_{k\ell}}{\partial y^2}-\frac{(\triangle y)^3}{6}\frac{\partial^3\psi_{k\ell}}{\partial y^3}+\\ &+\frac{(\triangle y)^4}{24}\frac{\partial^4\bar{\psi}_{k,\ell-1,\ell}}{\partial y^4}-4\psi_{k\ell}\Big]-k^2\psi_{k\ell}\ .\end{aligned}$$

In our case $\triangle x = \triangle y = h$, hence

$$\Delta_h\psi-k^2\psi = \frac{1}{h^2}[h^2(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2})]+\frac{1}{h^2}\frac{h^4}{24}[\frac{\partial^4\overline{\psi}_{k,k+1,\ell}}{\partial x^4}+\frac{\partial^4\overline{\psi}_{k,k-1,\ell}}{\partial x^4}+$$
$$+\frac{\partial^4\psi_{k,\ell+1,\ell}}{\partial y^4}+\frac{\partial^4\overline{\psi}_{k,\ell-1,\ell}}{\partial y^4}]-k^2\psi_{k\ell}$$

and we get

$$\Delta_h\psi-k^2\psi = H(\psi)+\frac{h^2}{24}R_{k\ell}(\psi) \tag{1.10}$$

where

$$H(\psi) = \Delta\psi-k^2\psi \tag{1.11}$$

and

$$R_{k\ell}(\psi) = \frac{\partial^4\overline{\psi}_{k,k+1,\ell}}{\partial x^4}+\frac{\partial^4\overline{\psi}_{k,k-1,\ell}}{\partial x^4}+\frac{\partial^4\overline{\psi}_{k,\ell+1,\ell}}{\partial y^4}+\frac{\partial\overline{\psi}_{k,\ell-1,\ell}}{\partial y^4} \tag{1.12}$$

where $\partial^4\overline{\psi}_{k,k+1,\ell}/\partial x^4$ denotes the value of $\partial^4\psi/\partial x^4$ at the midpoint of the segment joining the points $(k,\ell)$ and $(k+1,\ell)$, similarly $\partial^4\overline{\psi}_{k,k-1,\ell}/\partial x^4$ the value of $\partial^4\psi/\partial x^4$ at a point between $(k,\ell)$ and $(k-1,\ell)$, as well as $\partial^4\psi_{k,\ell+1,\ell}/\partial y^4$ and $\partial\psi_{k,\ell-1,\ell}/\partial y^4$ denote the values of $\partial^4\psi/\partial y^4$ at points between $(k,\ell)$ and $(k,\ell+1)$ and $(k,\ell-1)$ and $(k,\ell)$, respectively.

Coming back to (1.8) and substituting (1.10), we get

$$T(\psi) = [\frac{\Delta}{\Delta t}-(\frac{\Delta}{\Delta y}\psi)\frac{\Delta}{\Delta x}+(\frac{\Delta}{\Delta x}\psi)\frac{\Delta}{\Delta y}][H(\psi)+\frac{h^2}{24}R_{k\ell}(\psi)$$
$$= \frac{\Delta}{\Delta t}H(\psi)-(\frac{\Delta}{\Delta y}\psi)\frac{\Delta}{\Delta x}H(\psi)+(\frac{\Delta}{\Delta x}\psi)\frac{\Delta}{\Delta y}H(\psi)+ \tag{1.13}$$
$$+\frac{\Delta}{\Delta t}(\frac{h^2}{24}R_{k\ell}(\psi))-(\frac{\Delta}{\Delta y}\psi)\frac{\Delta}{\Delta x}(\frac{h^2}{24}R_{k\ell}(\psi))+(\frac{\Delta}{\Delta x}\psi)\frac{\Delta}{\Delta y}(\frac{h^2}{24}R_{k\ell}(\psi)$$

Let us first evaluate $(\Delta/\Delta t)H(\psi)$, i.e.,

$$\frac{\Delta}{\Delta t}H(\psi) = \frac{H^{m+1}(\psi)-H^m(\psi)}{\Delta t} \tag{1.14}$$

where superscripts m+1 and m denote the value of H at the time steps m+1 and m respectively. Having in mind the above made as-

sumption we see that $H(\psi)$ as defined by (1.11) has bounded derivatives up to order 2, hence, applying Taylor's expansion, we shall get:

$$\frac{\triangle}{\triangle t}H(\psi) = \frac{1}{\triangle t}[H^m(\psi)+\triangle t H_t^m(\psi)+\frac{(\triangle t)^2}{2}\bar{H}_{tt}^m(\psi)-H^m(\psi)] .$$

Considering the first time step, i.e., letting $m = 1$

(1.15) $$\frac{\triangle}{\triangle t}H(\psi) = H_t(\psi)+\frac{\triangle t}{2}\bar{H}_{tt}(\psi) .$$

Next we evaluate $\triangle/\triangle x H(\psi)$ and $\triangle/\triangle y H(\psi)$ expanding it into Taylor's series with respect to $x$ and $y$ respectively, obtaining

(1.16) $$\frac{\triangle}{\triangle x}H(\psi) = H_x(\psi)-\frac{\triangle x}{2}\bar{H}_{xx}(\psi)$$

and

(1.17) $$\frac{\triangle}{\triangle y}H(\psi) = H_y(\psi)-\frac{\triangle y}{2}\bar{H}_{yy}(\psi) .$$

However, to estimate $T(\psi)$ we shall need some more estimates, as $(\triangle/\triangle x)\psi$ and $(\triangle/\triangle y)\psi$. Consider

$$\frac{\triangle}{\triangle y}\psi = \frac{\psi_{k,\ell+1}-\psi_{k,\ell-1}}{2\triangle y} = \psi_y(x,y^*,t) = \psi_y(x,y,t)+\varepsilon .$$

Assuming that $\psi_y$ satisfies a Lipschitz condition, i.e.,

$$|\psi_y(x,y^*,t)-\psi_y(x,y,t)| \le k_1|y^*-y| \le k_1\triangle y .$$

We see that $\varepsilon$ must be proportional to $k_1\triangle y$, i.e.,

$$|\varepsilon| \le k_1\triangle y .$$

Therefore

(1.18) $$\frac{\triangle}{\triangle y}\psi = \psi_y+k_1\triangle y$$

and similarly

(1.19) $$\frac{\Delta}{\Delta x}\psi = \psi_x + k_2 \Delta x \ .$$

Next we find

$$\frac{\Delta}{\Delta t}[\frac{h^2}{24}R_{k\ell}(\psi)] \ , \quad \frac{\Delta}{\Delta x}[\frac{h^2}{24}R_{k\ell}(\psi)] \ , \quad \frac{\Delta}{\Delta y}[\frac{h^2}{24}R_{k\ell}(\psi)]$$

$$\frac{\Delta}{\Delta t}[\frac{h^2}{24}R_{k\ell}(\psi)] = \frac{h^2}{24}\frac{1}{\Delta t}[R(x,y,t+\Delta t)-R(x,y,t)] \ .$$

If we assume that $R_{k\ell}(\psi)$, defined by (1.12), satisfies a Lipschitz condition with respect to all the variables (i.e., the fourth derivatives of $\psi$ satisfy a Lipschitz condition)

$$|R(x,y,t+\Delta t)-R(x,y,t)| \leq \ell_1 \Delta t$$
$$|R(x+\Delta x,y,t)-R(x,y,t)| \leq \ell_2 \Delta x$$
$$|R(x,y+\Delta y,t)-R(x,y,t)| \leq \ell_3 \Delta y \ .$$

Considering the above we shall get the following estimates:

(1.20) $$\left|\frac{\Delta}{\Delta t}[\frac{h^2}{24}R(\psi)]\right| \leq h^2 \ell_1^*$$

(1.21) $$\left|\frac{\Delta}{\Delta x}[\frac{h^2}{24}R(\psi)]\right| \leq h^2 \ell_2^*$$

(1.22) $$\left|\frac{\Delta}{\Delta y}[\frac{h^2}{24}R(\psi)]\right| \leq h^2 \ell_3^* \ .$$

Denoting the following parts in (1.13) by B and C

(1.23) $$B = \frac{\Delta}{\Delta t}H(\psi) - (\frac{\Delta}{\Delta y}\psi)\frac{\Delta}{\Delta x}H(\psi) + (\frac{\Delta}{\Delta x}\psi)\frac{\Delta}{\Delta y}H(\psi)$$

(1.24) $$C = \frac{\Delta}{\Delta t}[\frac{h^2}{24}R_{k\ell}(\psi)] - (\frac{\Delta}{\Delta y}\psi)\frac{\Delta}{\Delta x}[\frac{h^2}{24}R_{k\ell}(\psi)] + $$
$$+ (\frac{\Delta}{\Delta x}\psi)\frac{\Delta}{\Delta y}[\frac{h^2}{24}R_{k\ell}(\psi)]$$

from (1.13) we have

(1.25) $$T(\psi) = B+C$$

and returning to the original expression for $H(\psi)$, as indicated in (1.11), and substituting (1.15), (1.16) and (1.17), also (1.18) and (1.19), into (1.23), remembering that $\triangle x = \triangle y = h$, we get

$$B = (\triangle\psi-k^2\psi)_t + \frac{\triangle t}{2}\overline{(\triangle\psi-k^2\psi)}_{tt} - (\psi_y+k_1h)[(\triangle\psi-k^2\psi)_x - \frac{h}{2}\overline{(\triangle\psi-k^2\psi)}_{xx}] + (\psi_x+k_2h)[(\triangle\psi-k^2\psi)_y - \frac{h}{2}\overline{(\triangle\psi-k^2\psi)}_{yy}]$$

$$B = (\triangle\psi-k^2\psi)_t - \psi_y(\triangle\psi-k^2\psi)_x + \psi_x(\triangle\psi-k^2\psi)_y + \frac{\triangle t}{2}\overline{(\triangle\psi-k^2\psi)}_{tt}$$

(1.26) $$- k_1h(\triangle\psi-k^2\psi)_x + \psi_x(\triangle\psi-k^2\psi)_y + \frac{h}{2}\psi_y\overline{(\triangle\psi-k^2\psi)}_{xx} + \frac{k_1h^2}{2}\overline{(\triangle\psi-k^2\psi)}_{xx} - \frac{h}{2}\psi_x\overline{(\triangle\psi-k^2\psi)}_{yy} - k_2\frac{h^2}{2}\overline{(\triangle\psi-k^2\psi)}_{yy} .$$

As $\psi$ is bounded and, by previous assumption, has bounded derivatives up to the 4th order, we can introduce the following bounds for certain expressions in (1.26):

(1.27) $$|\psi_x| \le N_1 \ ; \ |\psi_y| \le N_2 \ ; \ |(\triangle\psi-k^2\psi)_x| \le N_3 \ ; \ |(\triangle\psi-k^2\psi)_y| \le N_4$$

(1.28) $$|\overline{(\triangle\psi-k^2\psi)}_{tt}| \le N_5 \ ; \ |\overline{(\triangle\psi-k^2\psi)}_{xx}| \le N_6 \ ; \ |\overline{(\triangle\psi-k^2\psi)}_{yy}| \le N_7 .$$

Considering (1) and (1.a) and the bounds in (1.27) and (1.28), we get the following estimate for (1.26):

$$|B| \le \frac{\triangle t}{2}N_5 + k_1hN_3 + k_2hN_4 + \frac{h}{2}N_2N_6 + \frac{k_1h^2}{2}N_6 + \frac{h}{2}N_1N_7 + k_2\frac{h^2}{2}N_7$$

or

(1.29) $$|B| \le M_o\triangle t + M_1h + M_2h^2$$

where

$$(1.30)\quad M_o = \frac{N_5}{2}\ ;\quad M_1 = k_1N_3+k_2N_4+\frac{N_2N_6}{2}+\frac{N_1N_7}{2}\ ;\quad M_2 = \frac{k_1N_6+k_2N_7}{2}$$

Similarly, using (1.20), (1.21) and (1.22), (1.18) and (1.19) and bounds for $\psi_x$ and $\psi_y$, we get the following estimate for C:

$$|C| \leq \left|\frac{\triangle}{\triangle t}\left[\frac{h^2}{24}R_{k\ell}(\psi)\right]\right| + \left|\left(\frac{\triangle}{\triangle y}\psi\right)\right|\left|\frac{\triangle}{\triangle x}\left[\frac{h^2}{24}R_{k\ell}(\psi)\right]\right| + \left|\frac{\triangle}{\triangle x}\psi\right|\left|\frac{\triangle}{\triangle y}\right.$$

$$\left.\cdot\left[\frac{h^2}{24}R_{k\ell}(\psi)\right]\right|$$

$$|C| \leq \ell_1^*h^2+|\psi_y+k_1h|\,\ell_3^*h^2+|\psi_x+k_2h|\,\ell_2^*h^2$$

$$|C| \leq h^2[\ell_1^*+N_2\ell_3^*+k_1h\ell_3^*+N_1\ell_2^*+k_2\ell_2^*h]$$

$$(1.31)\qquad |C| \leq h^2[\ell_1^*+N_2\ell_3^*+N_1\ell_2^*]+h^3[k_1\ell_3^*+k_2\ell_2^*]$$

$$|C| \leq M_2^1h^2+M_3h^3\ .$$

And finally by (1.25), (1.29) and (1.31), we get the following bound for the truncation error $T(\psi)$

$$|T(\psi)| \leq M_o\triangle t+M_1h+M_2h^2+M_2^1h^2+M_3h^3$$

or

$$(1.32)\qquad |T(\psi)| \leq M_o\triangle t+M_1h+O(h^2)\ .$$

2. <u>Estimate for the Helmholtzian (Initial Error $E^{(o)} = 0$)</u>

We shall now proceed to estimate the error of the Helmholtzian after the first time step assuming that the initial error is zero. We will designate by $\Psi$ the values of the solu-

tion of the difference equation which replaces (1) and, for the time being, we shall ignore the round-off error and assume that all the arithmetic steps involved in solving the difference equation are carried out with infinite precision.

Hence

$$\left[\frac{\Delta}{\Delta t}-\left(\frac{\Delta}{\Delta y}\Psi\right)\frac{\Delta}{\Delta x}+\left(\frac{\Delta}{\Delta x}\Psi\right)\frac{\Delta}{\Delta y}\right]\left(\Delta_h\Psi-k^2\Psi\right) = 0 \ . \tag{2.1}$$

From (1.8), we have

$$\left[\frac{\Delta}{\Delta t}-\left(\frac{\Delta}{\Delta y}\psi\right)\frac{\Delta}{\Delta x}+\left(\frac{\Delta}{\Delta x}\psi\right)\frac{\Delta}{\Delta y}\right]\left(\Delta_h\psi-k^2\psi\right) = T(\psi) \ . \tag{2.2}$$

If we add to both sides of (2.2) the following auxiliary expression:

$$\left[-\left(\frac{\Delta}{\Delta y}\Psi\right)\frac{\Delta}{\Delta x}+\left(\frac{\Delta}{\Delta x}\Psi\right)\frac{\Delta}{\Delta y}\right]\left(\Delta_h\psi-k^2\psi\right)$$

and rearrange the terms, we shall get:

$$\begin{aligned}\left[\frac{\Delta}{\Delta t}-\left(\frac{\Delta}{\Delta y}\Psi\right)\frac{\Delta}{\Delta x}+\left(\frac{\Delta}{\Delta x}\Psi\right)\frac{\Delta}{\Delta y}\right]\left(\Delta_h\psi-k^2\psi\right) &= T(\psi)+\left[\left(\frac{\Delta}{\Delta y}\psi-\frac{\Delta}{\Delta y}\Psi\right)\right. \\ &\quad \left.\cdot\frac{\Delta}{\Delta x}+\left(\frac{\Delta}{\Delta x}\Psi-\frac{\Delta}{\Delta x}\psi\right)\frac{\Delta}{\Delta y}\right]\left(\Delta_h\psi-k^2\psi\right) \ .\end{aligned} \tag{2.3}$$

By subtracting (2.1) from (2.3) and denoting the error by e, i.e., $\psi-\Psi = e$, we get

$$\begin{aligned}\left[\frac{\Delta}{\Delta t}-\left(\frac{\Delta}{\Delta y}\Psi\right)\frac{\Delta}{\Delta x}+\left(\frac{\Delta}{\Delta x}\Psi\right)\frac{\Delta}{\Delta y}\right]\left(\Delta_h e-k^2 e\right) &= T(\psi)+\left[\left(\frac{\Delta}{\Delta y}e\right)\frac{\Delta}{\Delta x}\ -\right. \\ &\quad \left.-\left(\frac{\Delta}{\Delta x}e\right)\frac{\Delta}{\Delta y}\right]\left(\Delta_h\psi-k^2\psi\right)\end{aligned} \tag{2.4}$$

or

$$\frac{\Delta}{\Delta t}(\Delta_h e-k^2 e) = T(\psi)+[(\frac{\Delta}{\Delta y}e)\frac{\Delta}{\Delta x}-(\frac{\Delta}{\Delta x}e)\frac{\Delta}{\Delta y}](\Delta_h \psi-k^2\psi) + \tag{2.5}$$
$$+[(\frac{\Delta}{\Delta y}\Psi)\frac{\Delta}{\Delta x}-(\frac{\Delta}{\Delta x}\Psi)\frac{\Delta}{\Delta y}](\Delta_h e-k^2 e)$$

and

$$\frac{(\Delta_h e-k^2 e)^{m+1}-(\Delta_h e-k^2 e)^m}{\Delta t} = T(\psi)+[(\frac{\Delta}{\Delta y}e^m)\frac{\Delta}{\Delta x}-(\frac{\Delta}{\Delta x}e^m)\frac{\Delta}{\Delta y}] \tag{2.6}$$
$$(\Delta_h \psi^m-k^2\psi^m)+[(\frac{\Delta}{\Delta y}\Psi^m)\frac{\Delta}{\Delta x}-(\frac{\Delta}{\Delta x}\Psi^m)\frac{\Delta}{\Delta y}]$$
$$(\Delta_h e-k^2 e)^m$$

where the superscripts m+1 and m denote, as before, the $m+1^{st}$ and the $m^{th}$ time steps, respectively.

For the first time step we have to take $m = 0$, and by assuming that there is no initial error, i.e., $e^o = 0$, and $(\Delta_h e-k^2 e)^o = 0$, as well as $\frac{\Delta}{\Delta y}e^o = 0$; $\frac{\Delta}{\Delta x}e^o = 0$, (2.6) becomes:

$$(\Delta_h e-k^2 e)^1 = \Delta t T(\psi)$$

and

$$|(\Delta_h e-k^2 e)^1| \leq \Delta t|T(\psi)| \ .$$

Using the estimate for $T(\psi)$, i.e., (1.32) of the previous chapter and for convenience denoting $(\Delta_h e-k^2 e)^1$ by $\Delta_h e^{(1)}-k^2 e^{(1)}$ we shall get

$$|\Delta_h e^{(1)}-k^2 e^{(1)}| \leq \Delta t[M_o\Delta t+M_1 h+O(h^2)] \tag{2.7}$$
$$|\Delta_h e^{(1)}-k^2 e^{(1)}| \leq \delta^{(1)}$$

where

$$\delta^{(1)} = M_o(\Delta t)^2+M_1\Delta th+O(h^3) \ . \tag{2.8}$$

In part II using (2.7) we establish the following estimates for $e^{(1)}$ and its difference quotients $\frac{\Delta}{\Delta x}e^{(1)}$, $\frac{\Delta}{\Delta y}e^{(1)}$:

$$(2.9)\quad |e^{(1)}| \leq \delta^{(1)} , \quad |\frac{\Delta}{\Delta x}e^{(1)}| \leq C_o\delta^{(1)}|\log h| ,$$

$$|\frac{\Delta}{\Delta y}e^{(1)}| \leq C_o\delta^{(1)}|\log h|$$

which we shall presently use in the following error analysis.

Let us now consider the error of the Helmholtzian after the second time step, i.e., m = 1. From (2.6) we shall then have the following expression:

$$\begin{aligned}\Delta_h e^{(2)}-k^2e^{(2)} = {} & \Delta tT(\psi)+\Delta t[(\frac{\Delta}{\Delta y}e^{(1)})\frac{\Delta}{\Delta x}-(\frac{\Delta}{\Delta x}e^{(1)})\frac{\Delta}{\Delta y}] \\ (2.10)\qquad & (\Delta_h\psi^{(1)}-k^2\psi^{(1)})+[1+\Delta t(\frac{\Delta}{\Delta y}\Psi^{(1)})\frac{\Delta}{\Delta x} - \\ & -\Delta t(\frac{\Delta}{\Delta x}\Psi^{(1)})\frac{\Delta}{\Delta y}](\Delta_h e^{(1)}-k^2e^{(1)}) .\end{aligned}$$

In (2.10) let us denote by $K^{(1)}$ and $L^{(1)}$ the following parts:

$$\begin{aligned}K^{(1)} = {} & \Delta t(\frac{\Delta}{\Delta y}e^{(1)})\frac{\Delta}{\Delta x}(\Delta_h\psi^{(1)}-k^2\psi^{(1)})-\Delta t(\frac{\Delta}{\Delta x}e^{(1)}) \\ (2.11)\qquad & \cdot\frac{\Delta}{\Delta y}(\Delta_h\psi^{(1)}-k^2\psi^{(1)})\end{aligned}$$

$$(2.12)\quad L^{(1)} = [1+\Delta t(\frac{\Delta}{\Delta y}\Psi^{(1)})\frac{\Delta}{\Delta x}-\Delta t(\frac{\Delta}{\Delta x}\Psi^{(1)})\frac{\Delta}{\Delta y}](\Delta_h e^{(1)}-k^2e^{(1)})$$

$$(2.13)\qquad \Delta_h e^{(2)}-k^2e^{(2)} = \Delta tT(\psi)+K^{(1)}+L^{(1)}$$

and

$$(2.14)\quad |\Delta_h e^{(2)}-k^2e^{(2)}| \leq \Delta t|T(\psi)|+|K^{(1)}|+|L^{(1)}| .$$

Let us first get the upper bound for $L^{(1)}$ and later for $K^{(1)}$, From (2.12) we have

$$L^{(1)} = (\Delta_h e^{(1)} - k^2 e^{(1)}) + \Delta t\left(\frac{\Delta}{\Delta y}\mathbb{I}^{(1)}\right)\frac{\Delta}{\Delta x}(\Delta_h e^{(1)} - k^2 e^{(1)}) -$$

$$- \Delta t\left(\frac{\Delta}{\Delta x}\mathbb{I}^{(1)}\right)\frac{\Delta}{\Delta y}(\Delta_h e^{(1)} - k^2 e^{(1)})$$

and if

$$\text{(2.15)} \qquad \begin{aligned} -\frac{\Delta}{\Delta y}\mathbb{I}^{(1)} &= U > 0 \\ \frac{\Delta}{\Delta x}\mathbb{I}^{(1)} &= V > 0 \end{aligned}$$

we take the difference quotients as proposed in (1.4) and obtain:

$$\text{(2.16)} \qquad \begin{aligned} L^{(1)} = {} & (\Delta_h e^{(1)} - k^2 e^{(1)})_{k\ell} + \left(\frac{\Delta}{\Delta y}\mathbb{I}^{(1)}\right)\frac{\Delta t}{h}[(\Delta_h e^{(1)} - k^2 e^{(1)})_{k\ell} \\ & - (\Delta_h e^{(1)} - k^2 e^{(1)})_{k-1,\ell}] - \left(\frac{\Delta}{\Delta x}\mathbb{I}^{(1)}\right)\frac{\Delta t}{h}[(\Delta_h e^{(1)} \\ & - k^2 e^{(1)})_{k\ell} - (\Delta_h e^{(1)} - k^2 e^{(1)})_{k,\ell-1}] \end{aligned}$$

$$\text{(2.17)} \qquad \begin{aligned} L^{(1)} = {} & [(1 + \frac{\Delta t}{h}(\frac{\Delta}{\Delta y}\mathbb{I}^{(1)}) - \frac{\Delta t}{h}(\frac{\Delta}{\Delta x}\mathbb{I}^{(1)})](\Delta_h e^{(1)} - k^2 e^{(1)})_{k\ell} \\ & - \frac{\Delta t}{h}(\frac{\Delta}{\Delta y}\mathbb{I}^{(1)})(\Delta_h e^{(1)} - k^2 e^{(1)})_{k-1,\ell} + \frac{\Delta t}{h}(\frac{\Delta}{\Delta x}\mathbb{I}^{(1)}) \\ & \cdot (\Delta_h e^{(1)} - k^2 e^{(1)})_{k,\ell-1} \, . \end{aligned}$$

Next we shall show that $\frac{\Delta}{\Delta y}\mathbb{I}^{(1)}$ and $\frac{\Delta}{\Delta x}\mathbb{I}^{(1)}$ are bounded. Recalling that

$$\mathbb{I}^{(1)} = \psi^{(1)} - e^{(1)}$$

we get

$$\frac{\Delta}{\Delta x}\Psi^{(1)} = \frac{\Delta}{\Delta x}\psi^{(1)} - \frac{\Delta}{\Delta x}e^{(1)}$$

(2.18)

$$\left|\frac{\Delta}{\Delta x}\Psi^{(1)}\right| \le \left|\frac{\Delta}{\Delta x}\psi^{(1)}\right| + \left|\frac{\Delta}{\Delta y}e^{(1)}\right|$$

and similarly

(2.19) $$\left|\frac{\Delta}{\Delta y}\Psi^{(1)}\right| \le \left|\frac{\Delta}{\Delta y}\psi^{(1)}\right| + \left|\frac{\Delta}{\Delta y}e^{(1)}\right| .$$

From (1.18) and (1.19) we obtain:

(2.20) $$\left|\frac{\Delta}{\Delta x}\psi^{(1)}\right| \le |\psi_x^{(1)}| + k_2 h$$

(2.21) $$\left|\frac{\Delta}{\Delta y}\psi^{(1)}\right| \le |\psi_y^{(1)}| + k_1 h .$$

At this point we shall not use the bounds for $\frac{\Delta}{\Delta x}e$ and $\frac{\Delta}{\Delta y}e$ mentioned in the beginning of this chapter in (2.9) but evaluate $\frac{\Delta}{\Delta x}e$ and $\frac{\Delta}{\Delta y}e$ as follows:

$$\frac{\Delta}{\Delta x}e^{(1)} = \frac{e^{(1)}_{k+1,\ell} - e^{(1)}_{k-1,\ell}}{2h}$$

$$\left|\frac{\Delta}{\Delta x}e^{(1)}\right| \le \frac{1}{2h}\left(|e^{(1)}_{k+1,\ell}| + |e^{(1)}_{k-1,\ell}|\right) .$$

Using the bound for $e^{(1)}$ from (2.9) we obtain:

$$\left|\frac{\Delta}{\Delta x}e^{(1)}\right| \le \frac{1}{2h}2\delta^{(1)} = \frac{\delta^{(1)}}{h}$$

and using (2.8) we get

(2.22) $$\left|\frac{\Delta}{\Delta x}e^{(1)}\right| \le O(h) .$$

Similarly, we would get

(2.23) $$\left|\frac{\Delta}{\Delta y}e^{(1)}\right| \le O(h) .$$

Combining (2.18), (2.20), (2.22) and the bound for $\psi_x$ from (1.27), we obtain:

(2.24) $$\left|\frac{\Delta}{\Delta x}\Psi^{(1)}\right| \leq N_1+O(h) .$$

Similarly, combining (2.19), (2.21), (2.23) and the bound for $\psi_y$ from (1.27), we get:

(2.25) $$\left|\frac{\Delta}{\Delta y}\Psi^{(1)}\right| \leq N_2+O(h) .$$

If in (2.17) we denote $\frac{\Delta t}{h}$ by $\lambda$ and impose the following restriction

(2.26) $$1-\lambda\left[\left(\frac{\Delta}{\Delta x}\Psi^{(1)}\right)-\left(\frac{\Delta}{\Delta y}\Psi^{(1)}\right)\right] \geq 0$$

or

(2.27) $$\lambda \leq \frac{1}{\max\left[\left|\frac{\Delta}{\Delta x}\Psi^{(1)}\right|+\left|\frac{\Delta}{\Delta y}\Psi^{(1)}\right|\right]}$$

then, together with (2.15), (2.24) and (2.25), the above restriction implies that (2.17) is an average formed with non-negative weights which add up to one. Therefore,

$$|L^{(1)}| \leq \max|\Delta_h e^{(1)}-k^2 e^{(1)}|$$

and consequently, since by (2.7) $|\Delta_h e^{(1)}-k^2 e^{(1)}| \leq \delta^{(1)}$ ,

(2.28) $$|L^{(1)}| \leq \delta^{(1)} .$$

If (2.15) is not satisfied, which means that coefficients of (2.17) are of different signs, we can always adjust the difference quotients taking either forward or backward differences, whichever is necessary to get a convex combination for (2.17). In all cases the restriction for $\lambda$ will be the same, which is (2.27). If we denote $\Delta_h e^{(1)}-k^2 e^{(1)}$ by $G^{(1)}_{k\ell}$, the above can be contracted as follows:

$$-\frac{\Delta}{\Delta y}\Psi^{(1)} = U > 0 \quad \text{implies} \quad \frac{\Delta}{\Delta x}G^{(1)}_{k\ell} = \frac{1}{h}[G^{(1)}_{k\ell}-G^{(1)}_{k-1,\ell}]$$

$$-\frac{\Delta}{\Delta y}\Psi^{(1)} = U < 0 \quad \text{implies} \quad \frac{\Delta}{\Delta x}G^{(1)}_{k\ell} = \frac{1}{h}[G^{(1)}_{k+1,\ell}-G^{(1)}_{k\ell}]$$

$$\frac{\Delta}{\Delta x}\Psi^{(1)} = V > 0 \quad \text{implies} \quad \frac{\Delta}{\Delta y}G^{(1)}_{k\ell} = \frac{1}{h}[G^{(1)}_{k\ell}-G^{(1)}_{k,\ell-1}]$$

$$\frac{\Delta}{\Delta x}\Psi^{(1)} = V < 0 \quad \text{implies} \quad \frac{\Delta}{\Delta y}G^{(1)}_{k\ell} = \frac{1}{h}[G^{(1)}_{k,\ell+1}-G^{(1)}_{k\ell}] \; .$$

Then (2.17) becomes:

$$\text{(2.29)} \quad L^{(1)} = [1-\lambda(|U|+|V|)]G^{(1)}_{k\ell}+\lambda|U|G_{k-\text{sign } U,\ell}+\lambda|V|G_{k,\ell-\text{sign } V}$$

where

$$\text{sign } g = \begin{cases} -1 & \text{if } g < 0 \\ \;\;0 & \text{if } g = 0 \\ +1 & \text{if } g > 0 \end{cases} \; .$$

To get an upper bound for $K^{(1)}$ we shall use the estimates for $e^{(1)}$, $\frac{\Delta}{\Delta x}e^{(1)}$, $\frac{\Delta}{\Delta y}e^{(1)}$ which will be established in part II and which were briefly stated in (2.9). They are of sufficient order to insure the convergence of the proposed finite difference scheme. By (2.11)

$$\text{(2.30)} \quad |K^{(1)}| \le \Delta t\left|\frac{\Delta}{\Delta y}e^{(1)}\right|\left|\frac{\Delta}{\Delta x}(\Delta_h\psi^{(1)}-k^2\psi^{(1)})\right|+$$

$$+\Delta t\left|\frac{\Delta}{\Delta x}e^{(1)}\right|\left|\frac{\Delta}{\Delta y}(\Delta_h\psi^{(1)}-k^2\psi^{(1)}\right| \; .$$

From (1.10) we have

$$\Delta_h\psi-k^2\psi = H(\psi)+\frac{h^2}{24}R_{k\ell}(\psi)$$

where $H(\psi)$ and $R_{k\ell}(\psi)$ were defined by (1.11) and (1.12), respectively.

Hence

$$(2.31)\qquad \frac{\Delta}{\Delta x}(\Delta_h\psi-k^2\psi) = \frac{\Delta}{\Delta x}H(\psi)+\frac{\Delta}{\Delta x}[\frac{h^2}{24}R_{k\ell}(\psi)]$$

and

$$(2.32)\qquad |\frac{\Delta}{\Delta x}(\Delta_h\psi-k^2\psi)| \leq |\frac{\Delta}{\Delta x}H(\psi)|+|\frac{\Delta}{\Delta x}[\frac{h^2}{24}R_{k\ell}(\psi)]$$

but by (1.11) and (1.16)

$$\frac{\Delta}{\Delta x}H(\psi) = (\Delta\psi-k^2\psi)_x-\frac{h}{2}\overline{(\Delta\psi-k^2\psi)}_{xx}$$

and using corresponding bounds from (1.27) and (1.28) we obtain:

$$(2.33)\qquad |\frac{\Delta}{\Delta x}H(\psi)| \leq N_3+\frac{h}{2}N_6 \ .$$

Substituting (2.33) and (1.21) into (2.32) we obtain:

$$(2.34)\qquad \begin{aligned} |\frac{\Delta}{\Delta x}(\Delta_h\psi-k^2\psi)| &\leq N_3+hN_6^*+\ell_2^*h^2 \\ |\frac{\Delta}{\Delta x}(\Delta_h\psi-k^2\psi)| &\leq N_3+0(h) \ . \end{aligned}$$

Similarly using (1.17), (1.11) and (1.22) and the corresponding bounds from (1.27) we get:

$$(2.35)\qquad \begin{aligned} |\frac{\Delta}{\Delta y}(\Delta_h\psi-k^2\psi)| &\leq N_4+hN_7^*+h^2\ell_3^* \\ |\frac{\Delta}{\Delta y}(\Delta_h\psi-k^2\psi)| &\leq N_4+0(h) \ . \end{aligned}$$

If we substitute (2.34), (2.35) and the necessary bounds from (2.9) into (2.30) and keep in mind (2.8), we shall obtain:

$$(2.36)'\qquad |K^{(1)}| \leq \Delta tC_o\delta^{(1)}|\log h|[N_3+0(h)]+\Delta tC_o\delta^{(1)}\cdot |\log h|[N_6+0(h)]$$

(2.36)" $$|K^{(1)}| \leq N_o\Delta t\delta^{(1)}|\log h|+O(h^3)$$

where $N_o = C_o(N_3+N_6)$.

If we consider (2.28), (2.36) and recall that $\Delta t|T(\psi)| \leq$ $\leq \delta^{(1)}$ then by (2.14) for the second time step we get:

(2.37)
$$|\Delta_h e^{(2)}-k^2 e^{(2)}| \leq \delta^{(1)}+N_o\Delta t\delta^{(1)}|\log h|+\delta^{(1)}$$
$$|\Delta_h e^{(2)}-k^2 e^{(2)}| \leq 2\delta^{(1)}+\Delta t\delta^{(1)}N_o|\log h| = \delta^{(2)}$$

(2.38) $$|\Delta_h e^{(2)}-k^2 e^{(2)}| \leq \delta^{(2)} .$$

It is obvious that the method used in part II to get the previously stated bounds for $e^{(1)}$, $\frac{\Delta}{\Delta x}e^{(1)}$ and $\frac{\Delta}{\Delta y}e^{(1)}$ from (2.7), can be applied to obtain from (2.38) the estimates for $e^{(2)}$, $\frac{\Delta}{\Delta x}e^{(2)}$ and $\frac{\Delta}{\Delta y}e^{(2)}$. That is, we shall obtain:

(2.39)
$$|e^{(2)}| \leq \delta^{(2)} , \quad |\frac{\Delta}{\Delta x}e^{(2)}| \leq C_o\delta^{(2)}|\log h|$$
$$\text{and} \quad |\frac{\Delta}{\Delta y}e^{(2)}| \leq C_o\delta^{(2)}|\log h| .$$

In a similar way as before from (2.6) we shall obtain:

(2.40) $$|\Delta_h e^{(3)}-k^2 e^{(3)}| \leq \Delta t|T(\psi)|+|K^{(2)}|+|L^{(2)}|$$

where this time we shall obtain the following corresponding bounds for $K^{(2)}$ and $L^{(2)}$

(2.41) $$|K^{(2)}| \leq N_o\Delta t\delta^{(2)}|\log h|+O(h^3)$$

and

(2.42) $$|L^{(2)}| \leq \max|\Delta_h e^{(2)}-k^2 e^{(2)}| \leq \delta^{(2)} .$$

If we combine (2.40), (2.41), (2.42) and recall that $\Delta t|T(\psi)| \leq \delta^{(1)}$ we obtain

$$(2.43) \quad |\Delta_h e^{(3)} - k^2 e^{(3)}| \le \delta^{(1)} + N_o \Delta t \delta^{(2)} |\log h| + \delta^{(2)} .$$

If we substitute into (2.43) for $\delta^{(2)}$ its value in terms of $\delta^{(1)}$, i.e., (2.37) and leave out the superscript for $\delta^{(1)}$, i.e., $\delta^{(1)} = \delta$, we find that

$$|\Delta_h e^{(3)} - k^2 e^{(3)}| \le \delta + N_o \Delta t (2\delta + \Delta t \delta N_o |\log h|) |\log h|$$
$$+ 2\delta + \Delta t \delta N_o |\log h|$$

$$(2.44) \quad |\Delta_h e^{(3)} - k^2 e^{(3)}| \le 3\delta + 3\Delta t \delta N_o |\log h| + \delta(\Delta t N_o |\log h|)^2$$
$$= \delta^{(3)}$$

$$|\Delta_h e^{(3)} - k^2 e^{(3)}| \le \delta^{(3)} .$$

From (2.44) in a similar way as before we shall get the following estimates

$$(2.45) \quad \begin{aligned} &|e^{(3)}| \le \delta^{(3)} , \quad \left|\frac{\Delta}{\Delta x} e^{(3)}\right| \le \delta^{(3)} C_o |\log h| , \\ &\left|\frac{\Delta}{\Delta y} e^{(3)}\right| \le \delta^{(3)} C_o |\log h| . \end{aligned}$$

The above bounds can be used to get from (2.6)

$$(2.46) \quad |\Delta_h e^{(4)} - k^2 e^{(4)}| \le \delta^{(4)}$$

where $\delta^{(4)} = 4\delta + 6\delta \Delta t N_o |\log h| + 4\delta(N_o \Delta t |\log h|)^2 + \delta(\Delta t N_o |\log h|)^3$. From (2.46) we shall get the bounds for $e^{(4)}$, $\frac{\Delta}{\Delta x} e^{(4)}$, $\frac{\Delta}{\Delta y} e^{(4)}$. Hence we can continue this process a finite number of times. It is obvious that after n time steps we shall get the following formula

$$(2.47) \quad \begin{aligned} |\Delta_h e^{(n)} - k^2 e^{(n)}| \le \delta \Big\{ &\binom{n}{1} + \binom{n}{2} N_o \Delta t |\log h| + \binom{n}{3} (N_o \Delta t \cdot \\ &\cdot |\log h|)^2 + \ldots + \binom{n}{n-1} (N_o \Delta t |\log h|)^{n-2} + \binom{n}{n} (N_o \Delta t \cdot \\ &\cdot |\log h|)^{n-1} \Big\} \end{aligned}$$

or

$$(2.48)\qquad |\Delta_h e^{(n)} - k^2 e^{(n)}| \le \delta\cdot\left\{\frac{(1+N_o\Delta t|\log h|)^n - 1}{N_o\Delta t|\log h|}\right\}.$$

At this point we shall use the Law of the Mean, i.e., consider $N_o\Delta t|\log h| = y$, hence $f(y) = [(1+y)^n - 1]/y$ is continuous for $y_1 \le y \le y_2$ and differentiable for $y_1 < y < y_2$; therefore we can write $f(y) = [(1+y)^n - 1]/y = [(1+y)^n - (1+0)^n]/y = n(1+\theta y)^{n-1}$ where $0 < \theta < 1$. Hence (2.48) becomes:

$$|\Delta_h e^{(n)} - k^2 e^{(n)}| \le \delta n[1+\theta N_o\Delta t|\log h|]^{n-1} \le \delta n(1+N_o\Delta t|\log h|)^{n-1}$$

$$|\Delta_h e^{(n)} - k^2 e^{(n)}| \le \delta n\frac{(1+N_o\Delta t|\log h|)^n}{1+N_o\Delta t|\log h|} = \frac{\delta n\left(1+\frac{N_o\Delta t n|\log h|}{n}\right)^n}{1+N_o\Delta t|\log h|}$$

but $n\Delta t = T$, hence

$$|\Delta_h e^{(n)} - k^2 e^{(n)}| \le \frac{\delta n\left(1+\frac{N_o T|\log h|}{n}\right)^n}{1+N_o\Delta t|\log h|} \approx \frac{\delta n e^{-N_o T\log h}}{1-N_o\Delta t\log h}$$

(2.49)

$$|\Delta_h e^{(n)} - k^2 e^{(n)}| \le \frac{\delta n\, h^{-N_o T}}{1-N_o\Delta t\log h}.$$

Substituting for $\delta = \delta^{(1)}$ its value from (2.8) and recalling that $\lambda = \Delta t/h$ and $T = n\Delta t$ we shall obtain from (2.49) the following inequality:

$$|\Delta_h e^{(n)} - k^2 e^{(n)}| \le \frac{(M_o T\Delta t + M_1 T h)h^{-N_o T}}{1-N_o\Delta t\log h}$$

or

$$|\Delta_h e^{(n)} - k^2 e^{(n)}| \le \frac{(M_o^* + M_1)T h^{1-N_o T}}{1-N_o\lambda h\log h}$$

if we let $h \longrightarrow 0$ and $1-N_o T > 0$

$$(2.50)\qquad T < \frac{1}{N_o}$$

we obtain that

$$|\Delta_h e^{(n)} - k^2 e^{(n)}| \le K_o h^{1-N_oT} \tag{2.51}$$

hence if (2.50) is satisfied the Helmholtzian converges.

## 3. Estimate for the Helmholtzian (Initial Error $E^{(o)}$)

In the previous chapter we made the assumption that the initial error of the Helmholtzian is zero, i.e., $\Delta_h e^{(o)} - k^2 e^{(o)} = 0$. We shall now proceed with a convergence proof without the above assumption, i.e., we shall consider the initial error of the Helmholtzian to be given by

$$|\Delta_h e^{(o)} - k^2 e^{(o)}| \le E^{(o)} \ . \tag{3.1}$$

From (3.1) by previously mentioned and in part II in detail developed methods, we obtain

$$|e^{(o)}| \le E^{(o)} \ , \quad \left|\frac{\Delta}{\Delta x} e^{(o)}\right| \le C_o E^{(o)} |\log h| \ , \tag{3.2}$$

$$\left|\frac{\Delta}{\Delta y} e^{(o)}\right| \le C_o E^{(o)} |\log h| \ .$$

By (2.6) for m = 0, i.e., for $T = \Delta t$ we have

$$\begin{aligned}\Delta_h e^{(1)} - k^2 e^{(1)} = {} & \Delta t T(\psi) + \Delta t\left[\left(\frac{\Delta}{\Delta y} e^{(o)}\right)\frac{\Delta}{\Delta x} - \left(\frac{\Delta}{\Delta x} e^{(o)}\right)\frac{\Delta}{\Delta y}\right] \\ & (\Delta_h \psi^{(o)} - k^2 \psi^{(o)}) + \left[1 + \Delta t\left(\frac{\Delta}{\Delta y} \mathrm{I}^{(0)}\right)\frac{\Delta}{\Delta x}\right. \\ & \left. - \Delta t\left(\frac{\Delta}{\Delta x} \mathrm{I}^{(o)}\right)\frac{\Delta}{\Delta y}\right](\Delta_h e^{(o)} - k^2 e^{(o)})\end{aligned} \tag{3.3}$$

$$\Delta_h e^{(1)} - k^2 e^{(1)} = \Delta t T(\psi) + K^{(o)} + L^{(o)}$$

where

$$K^{(o)} = \triangle t\left(\frac{\triangle}{\triangle y}e^{(o)}\right)\frac{\triangle}{\triangle x}\left(\triangle_h\psi^{(o)}-k^2\psi^{(o)}\right)-\triangle t\left(\frac{\triangle}{\triangle x}e^{(o)}\right)\cdot\frac{\triangle}{\triangle y}\left(\triangle_h\psi^{(o)}-k^2\psi^{(o)}\right) \tag{3.4}$$

$$L^{(o)} = \left[1+\triangle t\left(\frac{\triangle}{\triangle y}\Psi^{(o)}\right)\frac{\triangle}{\triangle x}-\triangle t\left(\frac{\triangle}{\triangle x}\Psi^{(1)}\right)\frac{\triangle}{\triangle y}\right]\left(\triangle_h e^{(o)}-k^2e^{(o)}\right). \tag{3.5}$$

Using reasoning similar to that of chapter 2 we obtain the following upper bounds for $K^{(o)}$ and $L^{(o)}$, i.e.,

$$|K^{(o)}| \leq N_o\triangle t E^{(o)}|\log h| \tag{3.6}$$

$$|L^{(o)}| \leq \max|\triangle_h e^{(o)}-k^2e^{(o)}| \leq E^{(o)} \tag{3.7}$$

with the following restriction

$$\lambda = \frac{\triangle t}{h} \leq \frac{1}{\max\left\{\left|\frac{\triangle}{\triangle x}\Psi^{(o)}\right|+\left|\frac{\triangle}{\triangle y}\Psi^{(o)}\right|\right\}} . \tag{3.8}$$

From (3.3) using (3.6), (3.7) and the fact that $\triangle t|T(\psi)| \leq \delta$ we have

$$|\triangle_h e^{(1)}-k^2e^{(1)}| \leq \delta+E^{(o)}N_o\triangle t|\log h|+E^{(o)} = E^{(1)} \tag{3.9}$$

$$|\triangle_h e^{(1)}-k^2e^{(1)}| \leq E^{(1)} . \tag{3.10}$$

From (3.10) by previously employed methods we obtain:

$$|e^{(1)}| \leq E^{(1)} , \quad \left|\frac{\triangle}{\triangle x}e^{(1)}\right| \leq C_oE^{(1)}|\log h| , \tag{3.11}$$

$$\left|\frac{\triangle}{\triangle y}e^{(1)}\right| \leq C_oE^{(1)}|\log h| .$$

In a similar manner as before from (2.6) for $m = 1$, i.e., $T = 2\triangle t$ we get

(3.12) $$|\Delta_h e^{(2)}-k^2e^{(2)}| \le \Delta t|T(\psi)|+|K^{(1)}|+|L^{(1)}|$$

where this time $K^{(1)}$ and $L^{(1)}$ have the following upper bounds:

(3.13) $$|K^{(1)}| \le N_o\Delta tE^{(1)}|\log h|$$

(3.14) $$|L^{(1)}| \le \max|\Delta_h e^{(1)}-k^2e^{(1)}| \le E^{(1)}$$

and consequently

(3.15) $$|\Delta_h e^{(2)}-k^2e^{(2)}| \le \delta+N_o\Delta tE^{(1)}|\log h|+E^{(1)} .$$

Substituting the value of $E^{(1)}$ in terms of the initial error $E^{(o)}$ from (3.9) into (3.15) we obtain:

(3.16) $$|\Delta_h e^{(2)}-k^2e^{(2)}| \le 2\delta+\delta N_o\Delta t|\log h|+E^{(o)}+2E^{(o)}N_o\Delta t \cdot |\log h|+E_o(N_o\Delta t|\log h|)^2 = E^{(2)}$$

(3.17) $$|\Delta_h e^{(2)}-k^2e^{(2)}| \le E^{(2)}$$

and from the above by methods developed in part II,

(3.17) $$|e^{(2)}| \le E^{(2)} \; ; \quad \left|\frac{\Delta}{\Delta x}e^{(2)}\right| \le C_oE^{(2)}|\log h| \; ;$$
$$\left|\frac{\Delta}{\Delta y}e^{(2)}\right| \le C_oE^{(2)}|\log h| .$$

As a next step in the iteration process we get

(3.18) $$|\Delta_h e^{(3)}-k^2e^{(3)}| \le 3\delta+3\delta N_o\Delta t|\log h|+\delta(N_o\Delta t|\log h|)^2 + E^{(o)}+3E^{(o)}N_o\Delta t|\log h|+3E^{(o)}(N_o\Delta t \cdot |\log h|)^2+E^{(o)}(N_o\Delta t|\log h|)^3 = E^{(3)} .$$

It is not difficult to see that continuing the above iteration process a finite number of times, i.e., for $T = n\Delta t$, we shall get the following formula governing the error of the

Helmholtzian:

$$|\Delta_h e^{(n)}-k^2e^{(n)}| \leq \delta[\binom{n}{1}+\binom{n}{2}N_o\Delta t|\log h|+\binom{n}{3}(N_o\Delta t|\log h|)^2 + +\ldots+\binom{n}{n}(N_o\Delta t|\log h|)^{n-1}]+E^{(o)}[1+\binom{n}{1}N_o\Delta t \cdot \cdot|\log h|+\binom{n}{2}(N_o\Delta t|\log h|)^2+\ldots+\binom{n}{n}(N_o\Delta t \cdot \cdot|\log h|)^n]$$

or

$$(3.19)\quad |\Delta_h e^{(n)}-k^2e^{(n)}| \leq \delta\frac{(1+N_o\Delta t|\log h|)^n-1}{N_o\Delta t|\log h|}+E^{(o)}(1+N_o\Delta t \cdot \cdot |\log h|)^n .$$

The first part of (3.19) which we shall denote by P is identical with the right side of (2.48) and its estimate is given in (2.49), i.e.,

$$(3.20)\quad P = \delta\frac{(1+N_o\Delta t|\log h|)^n-1}{N_o\Delta t|\log h|} \leq \frac{\delta n h^{-N_oT}}{1-N_o\Delta t \log h} .$$

We shall denote the second part of (3.19) by S, i.e.,

$$S = E^{(o)}(1+N_o\Delta t|\log h|)^n = E^{(o)}(1+\frac{N_o\Delta t n|\log h|}{n})^n$$

but $n\Delta t = T$, and

$$S = E^{(o)}(1+\frac{N_oT|\log h|}{n})^n \approx E^{(o)}e^{-N_oT \log h}$$

hence

$$(3.21)\quad S \leq E^{(o)}h^{-N_oT} .$$

But

$$|\Delta_h e^{(n)}-k^2e^{(n)}| \leq P+S .$$

Therefore by (3.20) and (3.21) we obtain

$$|\Delta_h e^{(n)} - k^2 e^{(n)}| \leq \frac{\delta n h^{-N_oT}}{1-N_o\Delta t \log h} + E^{(o)} h^{-N_oT} .$$

Substituting the value of $\delta$ from (2.8) we obtain

$$|\Delta_h e^{(n)} - k^2 e^{(n)}| \leq \frac{(M_o\Delta t + M_1 h) T h^{-N_oT}}{1-N_o\Delta t \log h} + E^{(o)} h^{-N_oT}$$

but $\Delta t = \lambda h$, hence

$$|\Delta_h e^{(n)} - k^2 e^{(n)}| \leq \frac{(M_o\lambda + M_1) T h^{1-N_oT}}{1-N_o\lambda h \log h} + \frac{E^{(o)} h^{1-N_oT}}{h}$$

(3.22)

$$|\Delta_h e^{(n)} - k^2 e^{(n)}| \leq h^{1-N_oT} \left( \frac{K_o}{1-N_o\lambda h \log h} + \frac{E^{(o)}}{h} \right) .$$

If $E^{(o)} = O(h)$ and $T < 1/N_o$ then the error of the Helmholtzian $\longrightarrow 0$ as $h \longrightarrow 0$, and the growth of the error is governed by the above formula, or

(3.23) $$|\Delta_h e^{(T)} - k^2 e^{(T)}| \leq K_o h^{1-N_oT}$$

where (T) is used instead of (n) to indicate the reached time. By methods developed in part II we shall get

$$|e^{(T)}| \leq K_o h^{1-N_oT} .$$

## 4. Estimate for the Helmholtzian Extending the Total Time Beyond Previous Restrictions

In the previous chapter our formula governing the error was valid under the restriction that total time $T \leq 1/N_o$ and that the initial error $E^{(o)} = O(h)$. Our next object is to show that it is possible from previous results to develop a formula

which would show us that our proposed finite difference scheme converges even for a larger time than $T \leq 1/N_o$. To do so we shall proceed as follows. Consider that we reached the time $T \leq 1/N_o$ which from now on will be denoted by $T_1$ and called the first "time layer." In the previous chapter the respective error estimates for the Helmholtzian and the error itself were given by

$$(4.1) \quad |\Delta_h e^{(T_1)} - k^2 e^{(T_1)}| \leq h_1^{1-N_oT_1}(C_1T_1 + \frac{E^{(o)}}{h_1}) = E^{(T_1)}$$

$$(4.2) \quad |e^{(T_1)}| \leq h_1^{1-N_oT_1}(C_1T_1 + \frac{E^{(o)}}{h_1})$$

where $h_1$ denotes the width of the interval $\Delta x = \Delta y = h_1$ for the first "time layer." The above formulas are valid under the restriction that $T_1 \leq 1/N_o$ and $E^{(o)} = O(h_1)$.

Let us consider $E^{(T_1)}$ as our initial error. By previously discussed iterative error analysis we shall obtain the following error estimate for the second "time layer" with interval size $\Delta x = \Delta y = h_2$

$$(4.3) \quad |\Delta_h e^{(T_1+T_2)} - k^2 e^{(T_1+T_2)}| \leq h_2^{1-N_oT_2}(C_1T_2 + \frac{E^{(T_1)}}{h_1})$$

and

$$(4.4) \quad |e^{(T_1+T_2)}| \leq h_2^{1-N_oT_2}(C_1T_2 + \frac{E^{(T_1)}}{h_2}) .$$

From the above formulas it is obvious that $E^{(T_1)}$ has to be of order $h_2$, i.e., $E^{(T_1)} = O(h_2)$ and $T_2 \leq 1/N_o$. At this point we shall establish a relation between $h_2$ and $h_1$ and find an error estimate for the Helmholtzian at the endpoint of the second "time layer" in terms of the interval $h_1$. It is not difficult

to see from (4.1) that for $E^{(T_1)}$ to be of order $h_2$, taking into account that $E^{(o)} = O(h_1)$, the relation we are looking for is:

$$h_1^{1-N_oT_1} = h_2 \; . \tag{4.5}$$

From (4.3) and (4.1) we obtain

$$|\Delta_h e^{(T_1+T_2)} - k^2 e^{(T_1+T_2)}| \leq C_2T_2h_2^{1-N_oT_2} + \frac{h_2^{1-N_oT_2}}{h_2}[C_1T_1h_1^{1-N_oT_1} + \frac{E^{(o)}h_1^{1-N_oT_1}}{h_1}]$$

or

$$|\Delta_h e^{(T_1+T_2)} - k^2 e^{(T_1+T_2)}| \leq C_1T_2h_2^{1-N_oT_2} + \frac{C_1T_1h_2^{1-N_oT_2}h_1^{1-N_oT_1}}{h_2} + \frac{E^{(o)}h_2^{1-N_oT_2}h_1^{1-N_oT_1}}{h_1h_2} \; . \tag{4.6}$$

By (4.5)

$$h_2^{1-N_oT_2} = h_1^{(1-N_oT_1)(1-N_oT_2)} \; . \tag{4.7}$$

Substituting (4.5) and (4.7) into (4.6) we obtain

$$|\Delta_h e^{(T_1+T_2)} - k^2 e^{(T_1+T_2)}| \leq h_1^{(1-N_oT_1)(1-N_oT_2)}[C_1(T_1+T_2) + \frac{E^{(o)}}{h_1}] = E^{(T_2)} \tag{4.8}$$

and

$$|e^{(T_1+T_2)}| \leq E^{(T_2)} \; . \tag{4.9}$$

To get an estimate at the end of the third "time layer" we consider the initial error to be $E^{(T_2)}$ and, proceeding in a similar manner as before, we obtain the following estimate for the error of the Helmholtzian:

$$|\Delta_h e^{(T_1+T_2+T_3)} - k^2 e^{(T_1+T_2+T_3)}| \leq h_3^{1-N_oT_3}(C_1T_3 + \frac{E^{(T_2)}}{h_3}) \tag{4.10}$$

where $h_3$ denotes the width of the interval $\triangle x = \triangle y = h_3$ for the third "time layer." It is clear that $E^{(T_2)}$ has to be of order $h_3$ and $T_3 \leq 1/N_o$. 'or this purpose using (4.8) and (4.5) we establish the following relations between $h_3$ and $h_2$, and between $h_3$ and $h_1$, respectively:

$$h_3 = h_2^{1-N_oT_2} = h_1^{(1-N_oT_1)(1-N_oT_2)} \tag{4.11}$$

and

$$h_3^{1-N_oT_3} = h_1^{(1-N_oT_1)(1-N_oT_2)(1-N_oT_3)} . \tag{4.12}$$

Using (4.8) and (4.11), (4.10) becomes:

$$\begin{aligned} |\triangle_h e^{(T_1+T_2+T_3)} - k^2 e^{(T_1+T_2+T_3)}| &\leq h_1^{(1-N_oT_1)(1-N_oT_2)(1-N_oT_3)} \\ &\quad \cdot [C_1(T_1+T_2+T_3) + \frac{E^{(o)}}{h_1}] \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} |e^{(T_1+T_2+T_3)}| \leq h_1^{(1-N_oT_1)(1-N_oT_2)(1+N_oT_3)} [C_1(T_1+T_2+T_3) + \\ + \frac{E^{(o)}}{h_1}] . \end{aligned} \tag{4.14}$$

If we continue the above process a finite number of times, suppose k times, we shall obtain the following error estimate for the Helmholtzian at the end of the $k^{th}$ "time layer"

$$\begin{aligned} |\triangle_h e^{(T_1+T_2+\dots+T_k)} - k^2 e^{(T_1+T_2+\dots+T_k)}| \\ \leq h_1^{(1-N_oT_1)(1-N_oT_2)\dots(1-N_oT_k)} \\ \cdot [(T_1+T_2+\dots+T_k)C_1 + \frac{E^{(o)}}{h_1}] \end{aligned} \tag{4.15}$$

with the restriction that for each "time layer" $T_i \leq 1/N_o$ and $E^{(o)} = O(h_1)$. From (4.15)

$$(4.16)\quad |e^{(T_1+T_2+\dots+T_k)}| \le h_1^{(1-N_oT_1)(1-N_oT_2)\dots(1-N_oT_k)} \cdot [(T_1+\dots+T_k)C_1+\frac{E^{(o)}}{h_1}] \ .$$

If we take the restrictions on T for each "time layer"

$$T_1 \le \frac{1}{N_o}$$

$$T_2 \le \frac{1}{N_o}$$

$$\vdots$$

$$T_k \le \frac{1}{N_o}$$

and add the inequalities, we get

$$T_1+T_2+\dots+T_k \le \frac{k}{N_o} \ .$$

From the above inequality we see that it would be convenient to choose equal time steps for each layer, i.e., $T_1 = T_2 = \dots = T_k = T$, hence, if we denote the total time for all "time layers" by $\tilde{T}$, we obtain $\tilde{T} = kT$ and (4.15) becomes

$$(4.17)\quad |\Delta_h e^{(\tilde{T})} - k^2 e^{(\tilde{T})}| \le h_1^{(1-N_oT)^k}(\tilde{T}C_1+\frac{E^{(o)}}{h_1})$$

but

$$h_1^{(1-N_oT)^k} = h_1^{(1-(kTN_o/k))^k} \approx h_1^{e^{-\tilde{T}N_o}}$$

hence

$$|\Delta_h e^{(\tilde{T})} - k^2 e^{(\tilde{T})}| \le h_1^{e^{-\tilde{T}N_o}}(\tilde{T}C_1+\frac{E^{(o)}}{h_1})$$

and

$$|e^{(\tilde{T})}| \le h_1^{e^{-\tilde{T}N_o}}(K_o^*+\frac{E^{(o)}}{h_1})$$

where

$$K_o^* = \tilde{T}C_1 \ .$$

If we choose $E^{(o)} = O(h_1)$ then as $h_1 \longrightarrow 0$, $|e^{(\tilde{T})}| \longrightarrow 0$ (be-

cause $e^{-\tilde{T}N_o}$ is a positive number and $h_1^{e^{-\tilde{T}N_o}} \longrightarrow 0$ as $h_1 \longrightarrow 0$).
Hence our scheme is convergent for $\tilde{T} \leq \frac{k}{N_o}$.

## 5. Round-Off Error

In the previous error analysis we assumed that we can solve the finite difference equation (2.1) with infinite precision, i.e., the effect of the round-off has been ignored. In practice, however, we compute the solution of (2.1) rounded-off to a certain number of decimal places. If we keep the number of decimal places fixed and decrease the mesh width we could not expect convergence. We shall show that the scheme is convergent if the round-off is of order $O(h^4)$. Let us denote by $\overline{U}$ the quantity U after round-off, i.e., $\overline{U} = U+O(h^4)$.

It is not difficult to see that in solving (2.1) for the Helmholtzian we must require that the round-off be of order $O(h^2)$, i.e., $\overline{\mathcal{H}}^{m+1} = \mathcal{H}^{m+1}+O(h^2)$ where $\mathcal{H} = \Delta_h\Psi - k^2\Psi$. Consequently we establish that the round-off for $\Psi$ itself should be of order $O(h^4)$, i.e., $\overline{\Psi}^{m+1} = \Psi^{m+1}+O(h^4)$.

## Part II

## 6. Estimate for the Error $e = \psi - \Psi$

To find an estimate for $e^{(1)}$ from (2.7) we shall employ a special method given in [7] which involves the construction of an auxiliary function and is called the method of majorants. However, before proceeding with the above mentioned method we first need to prove a few theorems.

Theorem 1. If quantities $v_{k\ell}$ at all interior points of the grid domain $G_h$ satisfy: $\Delta_h v - k^2 v \leq 0$ and at the boundary points $\bar{v} \geq 0$, then $v \geq 0$ at all interior points.

Proof. Considering $\Delta_h v - k^2 v$ defined as before by (1.9) we assume the contrary, i.e., that at some points in the interior of $G_h$ the function v assumes negative values. Since v is non-negative on the boundary there would be an interior point at which v would assume its negative minimum, i.e., $v = \alpha < 0$ and at least one neighboring point at which $v > \alpha$, i.e.,

$$v_{k\ell} \leq v_{k+1,\ell}$$
$$v_{k\ell} \leq v_{k-1,\ell}$$
$$v_{k\ell} \leq v_{k,\ell+1}$$
$$v_{k\ell} \leq v_{k,\ell-1} \; ;$$

then it is obvious that in at least one of the inequalities the equality sign does not hold, hence after adding these inequalities and taking the above into consideration we obtain

$$v_{k+1,\ell} + v_{k-1,\ell} + v_{k,\ell+1} + v_{k,\ell-1} - 4v_{k\ell} > 0$$

and

$$-h^2k^2 v_{k\ell} \geq 0$$

because by assumption $v_{k\ell} \leq 0$ and adding once more we get

$$\Delta_h v - k^2 v = \frac{v_{k+1,\ell} + v_{k-1,\ell} + v_{k,\ell+1} + v_{k,\ell-1} - 4v_{k\ell} - k^2 h^2 v_{k\ell}}{h^2} > 0$$

which contradicts our assumption that $\Delta_h v - k^2 v \leq 0$. Hence we see that our assumption about v taking negative values in the interior, which lead us to this contradiction, must be false, and therefore we conclude that $v \geq 0$ at all interior points of $G_h$.

<u>Theorem 2</u>. If quantities $v_{k\ell}$ and $V_{k\ell}$ at all interior points of the grid domain $G_h$ satisfy the following inequality: $\Delta_h V - k^2 V \leq -|\Delta_h v - k^2 v|$ and at the boundary points $\bar{V}_{k\ell} \geq |\bar{v}_{k\ell}|$ then $V_{k\ell} \geq |v_{k\ell}|$ at all interior points of the grid domain $G_h$.

<u>Proof</u>. The above theorem follows directly from Theorem 1 if we observe that the inequality

$$\Delta_h V - k^2 V \leq -|\Delta_h v - k^2 v|$$

is equivalent to

$$\Delta_h V - k^2 V \leq \Delta_h v - k^2 v \quad \text{and} \quad \Delta_h V - k^2 V \leq -(\Delta_h v - k^2 v)$$

or

$$\Delta_h (V-v) - k^2 (V-v) \leq 0 \quad \text{and} \quad \Delta_h (V+v) - k^2 (V+v) \leq 0 \tag{6.1}$$

and that on the boundary

$$\bar{V}_{k\ell} - \bar{v}_{k\ell} \geq 0 \qquad \bar{V}_{k\ell} + \bar{v}_{k\ell} \geq 0 \; . \tag{6.2}$$

Considering (6.1) and (6.2) we apply Theorem 1 to quantities V+v and V-v and conclude that at all interior points of $G_h$

$$V+v \geq 0 \quad \text{and} \quad V-v \geq 0$$

or

$$|v| \leq V \; . \tag{6.3}$$

To estimate the error $e^{(1)}$ from $|\triangle_h e^{(1)} - k^2 e^{(1)}| \le \delta$ we shall employ the above mentioned method of majorants whose main idea is to construct a bounded function $z(x,y)$ which in $G_h$ would satisfy the following inequality $-(\triangle_h z - k^2 z) \ge \delta$ and get the estimate applying theorem 2.

We consider our grid domain $G_h$ to be a square $|x| \le 1$, $|y| \le 1$ and circumscribe around it a circle of radius $r = \sqrt{2}$ and center (0,0). We construct the following function

$$z(x,y) = \beta[1 - \frac{x^2}{2} - \frac{y^2}{2}] \tag{6.4}$$

and apply the operator $(\triangle_h - k^2)$, i.e.,

$$\triangle_h z - k^2 z = H(z) - R_{k\ell}(z) \tag{6.4a}$$

where $H(z)$ and $R_{k\ell}(z)$ are defined by (1.11) and (1.12).

$$H(z) = z_{xx} + z_{yy} - k^2 z$$

$$H(z) = -\beta - \beta - k^2\beta[1 - \frac{x^2}{2} - \frac{y^2}{2}] = -2\beta - \beta k^2[1 - \frac{x^2}{2} - \frac{y^2}{2}] .$$

But $R_{k\ell}$ in our case reduces to zero, because it involves derivatives of order higher than 2, i.e., $R_{k\ell}(z) = 0$. Hence (6.4a) becomes:

$$\triangle_h z - k^2 z = -2\beta[1 + \frac{k^2}{2}(1 - \frac{x^2}{2} - \frac{y^2}{2})]$$

$$-(\triangle_h z - k^2 z) = 2\beta[1 + \frac{k^2}{2}(1 - \frac{x^2}{2} - \frac{y^2}{2})]$$

but the factor of $k^2/2$ never becomes bigger than one and it is $\ge 0$ in $G_h$, hence we conclude

$$-(\triangle_h z - k^2 z) \ge 2\beta$$

and to achieve that $-(\triangle_h z - k^2 z) \ge \delta$ it is enough to choose $\beta$ to be equal to $\delta/2$, i.e., $\beta = \delta/2$, and

(6.5) $$-(\Delta_h z-k^2z) \geq \delta \;;$$

combining (2.7) and (6.5) we obtain

$$-(\Delta_h z-k^2z) \geq \delta \geq |\Delta_h e^{(1)}-k^2e^{(1)}|$$

at the interior points of $G_h$. On the boundary $z(x,y) = 0$ and if we assume that $e \geq 0$ on the boundary we are able to apply Theorem 2 and conclude that at all interior points

$$|e^{(1)}| \leq z$$

but

$$z= \frac{\delta}{2}[1-\frac{x^2}{2}-\frac{y^2}{2}] \leq \frac{\delta}{2} < \delta$$

and

(6.6) $$|e^{(1)}| \leq \delta \;.$$

## 7. Simplification of the Auxiliary Problem and "Discrete Potential Equation"

In this chapter we shall be concerned with the solution of the following problem

(7.1) $$\begin{aligned} \Delta_h e-k^2e &= f(P) && \text{for } P \text{ in } G_h \\ e &= 0 && \text{on the boundary of } G_h \end{aligned}$$

where $|f(P)| \leq \delta$.

Let us reduce the above problem as follows:

$$\Delta_h e = k^2e+f(P) = g(P)$$

by (6.6) $|e| \leq \delta$ hence $|g(P)| \leq c_o\delta$ where $c_o = k^2+1$, and we shall have the following problem

(7.2) $$\begin{aligned} \Delta_h e &= g(P) && P \,\varepsilon\, G_h \\ e &= 0 && \text{on the boundary .} \end{aligned}$$

We shall show that

$$e = e_1 + e_2$$

is a solution of the above problem if $e_1$ is a particular solution of

(7.4) $$\Delta_h e_1 = g(P)$$

and $e_2$ is a solution of the following boundary value problem

(7.5) $$\begin{aligned} \Delta_h e_2 &= 0 \quad P \varepsilon G_h \\ e_2 &= -e_1 \text{ on the boundary .} \end{aligned}$$

If $u(P,Q)$ is a function such that

(7.6) $$\Delta_h u(P,Q) = \begin{cases} \frac{1}{h^2} & P = Q \\ 0 & P \neq Q \end{cases}$$

then a particular solution of (7.4) can be written as

(7.7) $$e_1(P) = h^2 \sum_Q \sum u(P,Q)g(Q)$$

where the sum is taken over all grid points Q within the boundary of $G_h$. For if we operate $\Delta_h$ as defined by (1.3) on both sides of (7.7) with respect to the coordinates of P, and consider (7.6), we obtain

(7.8) $$\Delta_h e_1(P) = h^2 \sum_Q \sum \Delta_h u(P,Q)g(Q) = g(P) \text{ .}$$

For further analysis we shall need the estimates of (7.7) and its difference quotients, i.e.,

(7.9) $$|e_1(P)| \leq \max|g(Q)|h^2 \sum\sum |u(P,Q)|$$

(7.10) $$\left|\frac{\Delta}{\Delta x} e_1(P)\right| \leq \max|g(Q)|h^2 \sum\sum \left|\frac{\Delta}{\Delta x} u(P,Q)\right|$$

(7.11) $$\left|\frac{\Delta}{\Delta y} e_1(P)\right| \leq \max|g(Q)|h^2 \sum\sum \left|\frac{\Delta}{\Delta y} u(P,Q)\right| \text{ .}$$

To obtain the above estimates we actually have to estimate the following sums

$$h^2 \sum \sum |u(P,Q)| \tag{7.12}$$

$$h^2 \sum \sum \left|\frac{\Delta}{\Delta x} u(P,Q)\right| \tag{7.13}$$

$$h^2 \sum \sum \left|\frac{\Delta}{\Delta y} u(P,Q)\right| . \tag{7.14}$$

To do this we first have to obtain the bounds for the solution of (7.6) and its difference quotients.

A. Stöhr in reference [6], B. von der Pohl in [5] and S. L. Sobolev in [8] gave an extensive study of the solution of the following equation

$$\nabla u(\tilde{x},\tilde{y}) = \begin{cases} 1 & \tilde{x} = \tilde{y} = 0 \\ 0 & \text{otherwise} \end{cases} \tag{7.15}$$

where $\tilde{x},\tilde{y}$ are positive or negative integers including zero, and the symmetrical difference operator $\nabla$ is defined by

$$\nabla u(\tilde{x},\tilde{y}) = u(\tilde{x}+1,\tilde{y})+u(\tilde{x}-1,\tilde{y})+u(\tilde{x},\tilde{y}+1)+u(\tilde{x},\tilde{y}-1)-4u(\tilde{x},\tilde{y}) . \tag{7.16}$$

We shall use some of the results of B. von der Pohl [5] and A. Stöhr [6] to obtain the estimates for the above mentioned sums by showing an analogy between the solutions of (7.6) and (7.15).

B. von der Pohl in reference [5] verifies that the following function is a solution of (7.15)

$$u(\tilde{x},\tilde{y}) = \frac{1}{2\pi}\int_0^{2\pi} d\phi \frac{1}{2\pi}\int_0^{2\pi} d\psi \frac{1}{2}\cdot\frac{\sin^2(\tilde{x}\phi+\tilde{y}\psi)}{\sin^2\phi+\sin^2\psi} . \tag{7.17}$$

We shall now verify that

$$u(x,y;\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi} d\phi \frac{1}{2\pi}\int_0^{2\pi} d\psi \frac{1}{2}\cdot\frac{\sin^2[(x-\xi)\frac{\phi}{h}+(y-\eta)\frac{\psi}{h}]}{\sin^2\phi+\sin^2\psi} \tag{7.18}$$

is a solution of (7.6). Let us apply $\Delta_h$ on (7.18)

$$\Delta_h u = \frac{1}{4\pi^2}\int_0^{2\pi} d\phi \int_0^{2\pi} d\psi \frac{1}{2} \frac{\Delta_h \sin^2[(x-\xi)\frac{\phi}{h}+(y-\eta)\frac{\psi}{h}]}{\sin^2\phi+\sin^2\psi}. \tag{7.19}$$

It can be easily verified that

$$\Delta_h\left\{\sin^2[(x-\xi)\frac{\phi}{h}+(y-\eta)\frac{\psi}{h}]\right\} = \frac{2}{h^2}(\sin^2\phi+\sin^2\psi)\cdot \cos 2[(x-\xi)\frac{\phi}{h}+(y-\eta)\frac{\psi}{h}]. \tag{7.20}$$

If we substitute (7.20) into (7.19) we obtain:

$$\Delta_h u = \frac{1}{4\pi^2}\int_0^{2\pi} d\phi \int_0^{2\pi} d\psi \frac{1}{h^2}\cos 2[(x-\xi)\frac{\phi}{h}+(y-\eta)\frac{\psi}{h}]. \tag{7.21}$$

If $P = Q$, i.e., $(x,y) = (\xi,\eta)$, (7.21) becomes

$$\Delta_h u = \frac{1}{4\pi^2}\frac{1}{h^2}\int_0^{2\pi} d\phi \int_0^{2\pi} d\psi = \frac{1}{4\pi^2}\cdot\frac{1}{h^2}4\pi^2 = \frac{1}{h^2}$$

hence

$$\Delta_h u(P,Q) = \frac{1}{h^2} \quad \text{for} \quad P = Q,$$

and if $P \neq Q$ from (7.21) we get

$$\Delta_h u = \frac{1}{4\pi^2}\frac{1}{h^2}\int_0^{2\pi} d\phi \int_0^{2\pi} d\psi[\cos(x-\xi)\frac{\phi}{h}\cos(y-\eta)\frac{\psi}{h}-\sin(x-\xi)\frac{\phi}{h}\sin(y-\eta)\frac{\psi}{h}] \tag{7.22}$$

but we consider only the values at grid points, hence

$$x = \tilde{x}h \qquad \xi = \tilde{\xi}h$$
$$\text{and} \tag{7.23}$$
$$y = \tilde{y}h \qquad \eta = \tilde{\eta}h$$

and (7.22) becomes

$$\Delta_h u = \frac{1}{4\pi^2}\cdot\frac{1}{h^2}\left\{\int_0^{2\pi}\cos[(\tilde{x}-\tilde{\xi})\phi]d\phi\int_0^{2\pi}\cos[(\tilde{y}-\tilde{\eta})\psi]d\psi-\int_0^{2\pi}\sin[(\tilde{x}-\tilde{\xi})\phi)d\phi\cdot\right.$$

$$\left.\cdot\int_0^{2\pi}\sin[(\tilde{y}-\tilde{\eta})\psi]d\psi\right\} = \frac{1}{4\pi^2}\,\frac{1}{h^2}\,\frac{\sin(\tilde{x}-\tilde{\xi})\phi}{\tilde{x}-\tilde{\xi}}\Bigg|_0^{2\pi}\frac{\sin(\tilde{y}-\tilde{\eta})\psi}{\tilde{y}-\tilde{\eta}}\Bigg|_0^{2\pi} -$$

$$-\frac{1}{4\pi^2}\,\frac{1}{h^2}\,\frac{\cos(\tilde{x}-\tilde{\xi})\phi}{\tilde{x}-\tilde{\xi}}\Bigg|_0^{2\pi}\frac{\cos(\tilde{y}-\tilde{\eta})\psi}{\tilde{y}-\tilde{\eta}}\Bigg|_0^{2\pi}$$

but

$$\tilde{x} \neq \tilde{\xi}$$
$$\tilde{y} \neq \tilde{\eta}$$

hence

$$\Delta_h u = 0 - \frac{1}{4\pi h^2}\left[\frac{1}{\tilde{x}-\tilde{\xi}} - \frac{1}{\tilde{x}-\tilde{\xi}}\right] = 0$$

hence $\Delta_h u = 0$ for $P \neq Q$. We have shown that (7.18) is a solutuion of (7.6) and if into its right side we substitute for $(x,y)$ and $(\xi,\eta)$ its values from (7.23) we obtain for the solution of (7.6) the following expression:

$$(7.24)\qquad u(x,y;\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\int_0^{2\pi} d\psi\frac{1}{2}\,\frac{\sin^2[(\tilde{x}-\tilde{\xi})\phi+(\tilde{y}-\tilde{\eta})\psi]}{\sin^2\phi+\sin^2\psi}\quad .$$

Before proceeding with the estimates we shall establish a few properties of the above solution.

From (7.18) it is obvious that

$$(7.25)\qquad u(x,y;\xi,\eta) = u(\xi,\eta;x,y)\ .$$

Comparing (7.17) and (7.24) we see that

$$(7.26)\qquad u(x,y;0,0) = u(\tilde{x},\tilde{y};0,0) = u(\tilde{x},\tilde{y})\ .$$

From (7.24) it is obvious that

$$(7.27)\qquad u(0,0,0,0) = 0\ .$$

8. A Bound for the Non-Homogeneous Case

Next we seek an estimate for (7.12). It is obvious that considering (7.25) and (7.26) we can get it by estimating the following equivalent sum:

$$h^2 \sum_{\tilde{x},\tilde{y}\in G_h}\sum u(\tilde{x},\tilde{y}) \ . \tag{8.1}$$

The way is now prepared to employ the following bound for $u(\tilde{x},\tilde{y})$ given by A. Stöhr in reference [5]:

$$\left|u(\tilde{x},\tilde{y})-\frac{1}{2\pi}\log\sqrt{\tilde{x}^2+\tilde{y}^2}-\frac{3}{4\pi}\log 2-\frac{1}{2\pi}C\right| \leq \frac{M}{\tilde{x}^2+\tilde{y}^2} \ , \tag{8.2}$$

where $\tilde{x}^2+\tilde{y}^2 \neq 0$, C - Euler's constant, and M - a positive number independent of $\tilde{x}$ and $\tilde{y}$.

From (8.2) we obtain

$$|u(\tilde{x},\tilde{y})| \leq \left|\log\sqrt{\tilde{x}^2+\tilde{y}^2}\right|+K+\frac{M}{\tilde{x}^2+\tilde{y}^2} \tag{8.3}$$

where $K = \frac{3}{4\pi}\log 2+\frac{1}{2\pi}C$.

Having in mind (7.27), we should sum (8.3) over the following region $1 \leq \hat{x}^2+\hat{y}^2 = \hat{r}^2 \leq \tilde{R}^2$. However, let us subdivide the above region in two parts as follows:

$$1 \leq \tilde{r}^2 \leq (\tilde{R}')^2 \tag{8.4}$$

$$(\tilde{R}')^2 \leq \tilde{r}^2 \leq (\tilde{R})^2 \tag{8.5}$$

and take the following bounds for $u(\tilde{x},\tilde{y})$, respectively:

$$|u(\tilde{x},\tilde{y})| \leq \frac{1}{2\pi}\left|\log\sqrt{\tilde{x}^2+\tilde{y}^2}\right|+K+M \quad \text{for} \quad 1 \leq \tilde{r}^2 \leq (\tilde{R}')^2 \tag{8.6}$$

$$|u(\tilde{x},\tilde{y})| \leq \frac{1}{2\pi}\left|\log\sqrt{\tilde{x}^2+\tilde{y}^2}\right|+K+\frac{M}{\tilde{x}^2+\tilde{y}^2} \ \text{for} \ (\tilde{R}')^2 \leq \tilde{r}^2 \leq (\tilde{R})^2 \ . \tag{8.7}$$

Using (8.6) and (8.7) we obtain the following estimate for (8.1)

$$h^2 \sum\sum |u(\tilde{x},\tilde{y})| \le h^2 \sum_{1\le\tilde{r}\le R'}\sum [\frac{1}{2\pi}|\log \tilde{r}|+L] + \tag{8.8}$$
$$+ \sum_{\tilde{R}'\le\tilde{r}\le\tilde{R}}\sum [\frac{1}{2\pi}|\log \tilde{r}|+K+ \frac{M}{\tilde{r}^2}]$$

where $L = K+M$.

Returning to the original plane, i.e., taking $\tilde{r} = r/h$, we obtain the following subdivisions corresponding to (8.4) and (8.5), respectively:

$$h \le r \le R'$$
$$R' \le r \le R$$

and after choosing $R' = 1$ our range becomes

$$h \le r \le 1$$
$$1 \le r \le R \; .$$

Hence we obtain the following expression for (8.8).

$$h^2 \sum\sum |u(x,y)| \le h^2 \sum_{h\le r\le 1}\sum [\frac{1}{2\pi}|\log \frac{r}{h}|+L] +$$
$$+ h^2 \sum_{1\le r\le R}\sum [\frac{1}{2\pi}|\log \frac{r}{h}|+K+ \frac{Mh^2}{r^2}]$$

$$h^2 \sum\sum |u(x,y)| \le \frac{1}{2\pi} h^2 \sum_{h\le r\le 1}\sum |\log r| - \frac{1}{2\pi} h^2 \log h \sum_{h\le r\le 1}\sum 1 +$$
$$+ Lh^2 \sum_{h\le r\le 1}\sum 1 + \frac{1}{2\pi} h^2 \sum_{1\le r\le R}\sum |\log r| - \frac{1}{2\pi} h^2 \log h \cdot$$
$$\cdot \sum_{1\le r\le R}\sum 1 + Kh^2 \sum_{1\le r\le R}\sum 1 + h^2 M \sum_{1\le r\le R}\sum \frac{h^2}{r^2}$$

$$h^2 \sum\sum |u(x,y)| \le \frac{1}{2\pi} h^2 \sum_{h\le r\le 1}\sum |\log r| + \frac{1}{2\pi} h^2 |\log h| \sum_{h\le r\le R}\sum 1 +$$
$$+ Lh^2 \sum_{h\le r\le 1}\sum 1 + Kh^2 \sum_{1\le r\le R}\sum 1 + h^2 M \sum_{1\le r\le R}\sum \frac{h^2}{r^2} + \tag{8.10}$$
$$+ \frac{1}{2\pi} h^2 \sum_{1\le r\le R}\sum |\log r| \; .$$

We shall first approximate the following sum

(8.11) $$h^2 \sum_{h \le r \le 1}\sum |\log r| \;.$$

Let us now consider an h x h square in the subregion $h \le r \le 1$ of our grid domain $G_h$, and let us associate with each square a minimum value of $|\log r|$ at a grid point. It is obvious that we have to take the farthest point from the origin, i.e., in the first quadrant the point at the upper right hand corner of each h x h square; in the second quadrant -- the point at the upper left hand corner; in the third quadrant -- the point at the lower left hand corner, and in the fourth quandrant the point at the lower right hand corner, as indicated in Fig. 1.

Then it is obvious that

$$\iint_{\square_h^h} |\log r| dxdy \ge \min|\log r| h^2 \;.$$

Summing all the h x h squares of the region $h \le r \le 1$ we obtain

(8.12) $$\sum\sum\iint |\log r| dxdy \ge h^2 \sum\sum |\log r| \;.$$

The sum on the right hand side is the sum over all grid points except the points on the horizontal and vertical axes. Hence, to make the right-hand side of (8.12) the summation over all net points, we add to both sides of the above inequalities the following expression $\frac{4}{h}h^2 \log h + b_o h^2$ (where $4/h$ = number of points on the vertical and horizontal lines and $b_o$ a positive constant).

$$\iint_{h \le r \le 1} |\log r| dxdy + \frac{4}{h}h^2 \log h + b_o h^2 \ge h^2 \sum_{\text{all n.p.}}\sum |\log r|$$

$$\lim_{h\to 0}\Big[\int_0^{2\pi} d\theta\int_h^1 |\log r|r\,dr + 4h\log h + b_o h^2 \geq h^2 \sum_{\text{all}}\sum_{\text{n.p.}} |\log r|$$

$$\int_0^{2\pi} d\theta\int_0^1 |\log r|r\,dr \geq h^2 \sum_{\text{all}}\sum_{\text{n.p.}} |\log r|$$

(8.13) $$h^2 \sum_{\text{all}}\sum_{\text{n.p.}} |\log r| \leq \int_0^{2\pi} d\theta\int_0^1 |\log r|r\,dr \ .$$

Now we shall estimate the remaining sums in (8.10)

(8.14) $$\frac{1}{2\pi}h^2|\log h| \sum_{1\leq r\leq R}\sum 1 = \frac{1}{2\pi}h^2|\log h|\Big[\frac{4R}{h}\cdot\frac{R}{h}+\frac{4R}{h}\Big] = \frac{2R^2}{\pi}|\log h| + \frac{2R}{\pi}h|\log h|$$

(8.15) $$\frac{1}{2\pi}h^2|\log h| \sum_{1\leq r\leq R}\sum 1 = d_2|\log h| + d_2^* h|\log h|$$

(8.16) $$Kh^2 \sum_{1\leq r\leq R}\sum 1 = Kh^2\Big[\frac{4R^2}{h^2}+\frac{4R}{h}\Big] = b_o^* + b_1^* h$$

(8.17) $$h^2 M \sum_{1\leq r\leq R}\sum \frac{h^2}{r^2} \approx h^2 M \int_0^{2\pi} d\theta \int_1^R \frac{dr}{r} = d_o h^2$$

(8.18) $$h^2 \sum_{1\leq r\leq R}\sum |\log r| \approx \int_0^{2\pi} d\theta\int_1^R |\log r|r\,dr \ .$$

Hence by (8.13), (8.15), (8.16), (8.17) and (8.18) we obtain for (8.10)

$$h^2 \sum\sum |u(x,y)| \leq \int_0^{2\pi} d\theta\int_0^R |\log r|r\,dr) + d_2|\log h| + d_2^* h|\log h| + b_1^* h + b_o^* + d_o h^2$$

$$h^2 \sum\sum |u(x,y)| \leq 2\pi R^2\Big[\frac{\log R}{2} - \frac{1}{4}\Big] + b_o^* + d_2|\log h| + d_2^* h|\log h| + b_1^* h + d_o h^2 .$$

(8.19)

$$h^2 \sum\sum |u(x,y)| \leq d_1 + d_2|\log h| + d_2^* h|\log h| + b_1^* h + d_o h^2 \ .$$

Now we are ready to obtain a bound for $e_1(P)$. By (7.9)

(8.20) $$|e_1(P)| \leq \max|g(Q)|h^2 \sum\sum |u(P,Q)| \ .$$

But

$$g(P) = k^2e(P)+f(P)$$

hence

$$|g(P)| \leq k^2|e(P)|+|f(P)| \ .$$

By (1.6) and the fact that $|f(P)| \leq \delta$ we obtain

$$|g(P)| \leq c_o\delta \text{ where } c_o = k^2+1 \ .$$

Therefore considering the above bound for $g(P)$ and (8.19), from (8.20) we get

$$|e_1(P)| \leq c_o\delta[d_1+d_2|\log h|+d_2^*h|\log h|+b_1^*h+d_oh^2] \ .$$

Since by (2.8)

$$\delta = M_o(\triangle t)^2+M_1\triangle th+0(h^3)$$

we shall have

(8.21) $$|e_1(P)| \leq c_o^*\delta|\log h|+0(h^2) \ .$$

## 9. Another Bound for the Non-Homogeneous Case

In this chapter we shall find an estimate for (7.13). By a similar argument as in chapter 8, it is obvious that (7.13) can be approximated by estimating the following equivalent sum

(9.1) $$h^2 \sum\sum \frac{1}{h} \frac{\triangle}{\triangle\tilde{x}} u(\tilde{x},\tilde{y})$$

since it can be easily seen from (7.17) and (7.24) that

(9.2) $$\frac{\triangle}{\triangle x} u(x,y) = \frac{1}{h} \frac{\triangle}{\triangle\tilde{x}} u(\tilde{x},\tilde{y}) \ .$$

A. Stöhr's paper [6] gives an estimate for the above-mentioned

function, i.e.,

$$(9.3)\quad |u(\tilde{x}+1,\tilde{y})-u(\tilde{x},\tilde{y})| \leq \frac{1}{2\pi}\left|\log\sqrt{(\tilde{x}+1)^2+\tilde{y}^2}-\log\sqrt{\tilde{x}^2+\tilde{y}^2}\right|+ + \frac{M}{(\tilde{x}+1)^2+\tilde{y}^2}+\frac{M}{\tilde{x}^2+\tilde{y}^2}$$

for $\tilde{x}^2+\tilde{y}^2 = \tilde{r}^2 > 1$.

We shall subdivide our region in the following way:

$$2 \leq \tilde{r}^2 \leq (\tilde{R}')^2$$
$$(\tilde{R}')^2 \leq \tilde{r}^2 \leq (\tilde{R})^2 .$$

Since $\tilde{x}^2+\tilde{y}^2 > 1$, it is obvious that

$$\frac{M}{\tilde{x}^2+\tilde{y}^2}+\frac{M}{(\tilde{x}+1)^2+\tilde{y}^2} \leq \frac{M}{\sqrt{\tilde{x}^2+\tilde{y}^2}}+\frac{M}{\sqrt{(\tilde{x}+1)^2+\tilde{y}^2}}$$

and we shall take the following estimates for different regions:

$$(9.4)\quad |u(\tilde{x}+1,\tilde{y})-u(\tilde{x},\tilde{y})| \leq \frac{1}{2\pi}\left|\log\sqrt{(\tilde{x}+1)^2+\tilde{y}^2}-\log\sqrt{\tilde{x}^2+\tilde{y}^2}\right|+ + \frac{M}{\sqrt{\tilde{x}^2+\tilde{y}^2}}\quad\frac{M}{\sqrt{(\tilde{x}+1)^2+\tilde{y}^2}}$$

for $2 \leq \tilde{r}^2 \leq (\tilde{R}')^2$ and

$$(9.5)\quad |u(\tilde{x}+1,\tilde{y})-u(\tilde{x},\tilde{y})| \leq \frac{1}{2\pi}\left|\log\sqrt{(\tilde{x}+1)^2+\tilde{y}^2}-\log\sqrt{\tilde{x}^2+\tilde{y}^2}\right|+ + \frac{M}{(\tilde{x}+1)^2+\tilde{y}^2}+\frac{M}{\tilde{x}^2+\tilde{y}^2}$$

for $(\tilde{R}')^2 \leq \tilde{r}^2 \leq (\tilde{R})^2$.

Considering (9.2) as well as (9.4) we obtain for $\frac{\triangle}{\triangle x}u(x,y)$ the following estimate in $2 \leq \tilde{r} \leq (\tilde{R}')^2$

$$\left|\frac{\triangle}{\triangle x}u(x,y)\right| = \frac{1}{h}|u(\tilde{x}+1,\tilde{y}|-u(\tilde{x},\tilde{y})| \leq \frac{1}{h}\cdot\frac{1}{4\pi}\left|\log\left(1+\frac{2\tilde{x}+1}{\tilde{x}^2+\tilde{y}^2}\right)\right|+ + \frac{M}{\sqrt{\tilde{x}^2+\tilde{y}^2}}+\frac{M}{\sqrt{(\tilde{x}+1)^2+\tilde{y}^2}}$$

$$\left|\frac{\Delta}{\Delta x}u(x,y)\right| \le \frac{1}{h}\left\{\frac{1}{4h}\left|\frac{2\tilde{x}+1}{\tilde{x}^2+\tilde{y}^2}\right| + \frac{M}{\sqrt{\tilde{x}^2+\tilde{y}^2}} + \frac{M}{\sqrt{(\tilde{x}+1)^2+\tilde{y}^2}}\right\} .$$

Introducing polar coordinates, i.e., letting

$$\tilde{x} = \tilde{r}\cos\alpha$$

$$\tilde{y} = \tilde{r}\sin\alpha$$

we get

$$\left|\frac{\Delta}{\Delta x}u(x,y)\right| \le \frac{1}{h}\left\{\frac{1}{4\pi}\,\frac{2\tilde{r}\cos\alpha+1}{\tilde{r}^2} + \frac{M}{\tilde{r}} + \frac{M}{\sqrt{\tilde{r}^2+2\tilde{r}\cos\alpha+1}}\right\}$$

$$\le \frac{1}{h}\left\{\frac{1}{4\pi}\,\frac{2\tilde{r}+1}{\tilde{r}^2} + \frac{M}{\tilde{r}} + \frac{M}{\sqrt{\tilde{r}^2-2\tilde{r}+1}}\right\}$$

$$\left|\frac{\Delta}{\Delta x}u(x,y)\right| \le \frac{1}{h}\left\{\frac{1}{2\pi}\cdot\frac{1}{\tilde{r}} + \frac{1}{4\pi}\cdot\frac{1}{\tilde{r}^2} + \frac{M}{\tilde{r}} + \frac{M}{\tilde{r}-1}\right\}$$

but

$$\frac{1}{4\pi\tilde{r}^2} \le \frac{1}{4\pi\tilde{r}} \qquad \text{for} \qquad \tilde{r} > 1 \quad ,$$

hence

$$\left|\frac{\Delta}{\Delta x}u(x,y)\right| \le \frac{1}{h}\left\{\left(\frac{1}{2\pi} + \frac{1}{4\pi} + M\right)\frac{1}{\tilde{r}} + \frac{M}{\tilde{r}-1}\right\}$$

$$\left|\frac{\Delta}{\Delta x}u(x,y)\right| \le \frac{1}{h}\left[\frac{p_o}{\tilde{r}} + \frac{M}{\tilde{r}-1}\right] .$$

Returning to the original plane, i.e., letting $r = \tilde{r}h$ we obtain

(9.6) $$\left|\frac{\Delta}{\Delta x}u(x,y)\right| \le \frac{p_o}{r} + \frac{M}{r-h}$$

in $2h \le r \le 1$ if we choose $R' = 1$. As a next step we shall estimate

(9.7) $$h^2 \sum_{2h\le r\le 1}\sum \left|\frac{\Delta}{\Delta x}u(x,y)\right| \le h^2 \sum_{2h\le r\le 1}\sum \frac{p_o}{r} + h^2 \sum_{2h\le r\le 1}\sum \frac{M}{r-h} .$$

We shall first approximate the following sum:

$$h^2 \sum_{2h \leq r \leq 1}\sum \frac{p_o}{r} .$$

As before we consider an h x h square in the region $2h \leq r \leq 1$ and let associate with each square a minimum value of $P_o/r$. It is obvious that we have to take the farthest point from the origin and exactly as in chapter 8 we will obtain for every h x h square the following inequality:

$$\iint_{\square_h^h} \frac{p_o}{r} dxdy \geq \min\left(\frac{p_o}{r}\right) \cdot h^2 .$$

Adding the above inequalities for all h x h squares in the region $2h \leq r \leq 1$ we obtain

$$\sum\sum \iint_{\square} \frac{p_o}{r} dxdy \geq h^2 \sum\sum \frac{p_o}{r}$$

where the sum on the right hand side is over all the grid points of the region $2h \leq r \leq 1$ except the points on the horizontal and vertical lines. Hence if we add to both sides the (max value of $p_o/r$) x $h^2$ x (number of net points on vertical and horizontal lines) we obtain:

$$\int_0^{2\pi} d\theta \int_{2h}^{1} p_o dr + h^2 \frac{p_o}{2h} 4\left(\frac{1}{h}-1\right) \geq h^2 \sum_{\text{all n pts.}}\sum \frac{p_o}{r}$$

$$2\pi p_o(1-2h) + 2(p_o - p_o h) \geq h^2 \sum_{\text{all n pts.}}\sum \frac{p_o}{r}$$

or

$$h^2 \sum_{\text{all n.p.}}\sum \frac{p_o}{r} \leq m_o + n_o h . \tag{9.9}$$

As a next step we shall evaluate

(9.10) $$h^2 \sum_{2h \le r \le 1}\sum \frac{M}{r-h} .$$

As before we associate with each h x h square a minimum value of $M/(r-h)$ which will occur as before at the farthest point from the origin, and as previously we will obtain

$$\iint_{\square_h^h} \frac{M}{r-h} dxdy \geq \min\left(\frac{M}{r-h}\right) \times h^2 ;$$

again summing all such inequalities for all the squares we obtain

$$\sum\sum \iint \frac{M}{r-h} dxdy \geq h^2 \sum\sum \frac{M}{r-h}$$

but

$$\iint_{2h \le r \le 1} \frac{g_o dxdy}{r/2} \geq \sum\sum \iint \frac{M}{r-h} dxdy$$

hence

(9.11) $$\iint_{2h \le r \le 1} \frac{g_o dxdy}{r/2} \geq h^2 \sum\sum \frac{M}{r-h} .$$

The sum on the right of (9.11) is over all the grid points except the ones on the horizontal and vertical lines.

Adding to both sides of (9.11) the following expression: $h^2$ x (no. of points on horizontal and vertical lines) x max x $\left(\frac{M}{r-h}\right)$, we have

$$2\int_o^{2\pi} d\theta \int_{2h}^{1} g_o dr + h^2 \frac{M}{h}\left(\frac{1}{h}-1\right)4 \geq h^2 \sum_{2h \le r \le 1}\sum \frac{M}{r-h}$$

(9.12) $$h^2 \sum_{2h \le r \le 1}\sum \frac{M}{r-h} \leq m_o^* + n_o^* h .$$

Combining (9.9), (9.12) and (9.7) we obtain

$$h^2 \sum_{2h\le r\le 1}\sum \left|\frac{\Delta}{\Delta x}u(x,y)\right| \le m_1+n_1h \tag{9.13}$$

where $m_1, n_1$ are constants.

However, we need an estimate of (9.2) in the whole domain $0 \le r \le 1$, therefore we shall next find an estimate for the remaining part: $0 \le r \le 2h$. We estimate the difference quotients from numerical values of the function given by A. Stöhr as well as by B. van der Pohl in [6] and [5], respectively. Hence we consider the following sum

$$\begin{aligned} h^2 \sum_{0\le r\le 1}\sum \left|\frac{\Delta}{\Delta x}u(x,y)\right| &= h^2 \sum_{0\le r\le 2h}\sum \left|\frac{\Delta}{\Delta x}u(x,y)\right| + \\ &+ h^2 \sum_{2h\le r\le 1}\sum \left|\frac{\Delta}{\Delta x}u(x,y)\right| . \end{aligned} \tag{9.14}$$

The estimate for the second sum on the right hand side is given by (9.13). The first sum will consist of the following quotients in I$^{st}$, II$^{nd}$, III$^{rd}$ and IV$^{th}$ quadrants, respectively (see Fig. 1).

$$\frac{1}{h}[u(2h,0)-u(h,0)] = \frac{1}{h}[u(2,0)-u(1,0)] \tag{9.I$_a$}$$

$$\frac{1}{h}[u(2h,h)-u(h,h)] = \frac{1}{h}[u(2,1)-u(1,1)] \tag{9.I$_b$}$$

$$\frac{1}{h}[u(h,0)-u(0,0)] = \frac{1}{h}[u(1,0)-u(0,0)] \tag{9.I$_c$}$$

$$\frac{1}{h}[u(h,h)-u(0,h)] = \frac{1}{h}[u(1,1)-u(0,1)] \tag{9.I$_d$}$$

$$\frac{1}{h}[u(-2h,0)-u(-h,0)] = \frac{1}{h}[u(-2,0)-u(-1,0)] \tag{9.II$_a$}$$

$$\frac{1}{h}[u(-2h,h)-u(-h,h)] = \frac{1}{h}[u(-2,1)-u(-1,1)] \tag{9.II$_b$}$$

$$\frac{1}{h}[u(-h,0)-u(0,0)] = \frac{1}{h}[u(-1,0)-u(0,0)] \tag{9.II$_c$}$$

$$\frac{1}{h}[u(-h,h)-u(0,h)] = \frac{1}{h}[u(-1,1)-u(0,1)] \tag{9.II$_d$}$$

$$(9.\mathrm{III}_b)\qquad \frac{1}{h}[u(-2h,-h)-u(-h,-h)] = \frac{1}{h}[u(-2,-1)-u(-1,-1)]$$

$$(9.\mathrm{III}_d)\qquad \frac{1}{h}[u(-h,-h)-u(0,-h)] = \frac{1}{h}[u(-1,-1)-u(0,-1)]$$

$$(9.\mathrm{IV}_b)\qquad \frac{1}{h}[u(2h,-h)-u(h,-h)] = \frac{1}{h}[u(2,-1)-u(1,-1)]$$

$$(9.\mathrm{IV}_d)\qquad \frac{1}{h}[u(h,-h)-u(0,-h)] = \frac{1}{h}[u(1,-1)-u(0,-1)] \;.$$

A. Stöhr in [6] gives the following relations for $u(\tilde{x},\tilde{y})$:

$$(9.15)\qquad \begin{aligned} u(\tilde{x},\tilde{y}) &= u(\hat{x},-\tilde{y}) = u(-\hat{x},\tilde{y})-u(-\hat{x},-\tilde{y}) = u(\tilde{y},\tilde{x}) = u(-\tilde{y},\tilde{x}) \\ &= u(-\tilde{y},-\tilde{x}) = u(\tilde{y},-\tilde{x}) \;. \end{aligned}$$

Due to which the difference quotients only in the first quadrant have to be estimated, since

$$(9.\mathrm{I}_a) = (9.\mathrm{II}_a)$$

$$(9.\mathrm{I}_b) = (9.\mathrm{II}_b) = (9.\mathrm{III}_b) = (9.\mathrm{IV}_b)$$

$$(9.\mathrm{I}_c) = (9.\mathrm{II}_c)$$

$$(9.\mathrm{I}_d) = (9.\mathrm{II}_d) = (9.\mathrm{III}_d) = (9.\mathrm{IV}_d) \;.$$

From A. Stöhr's table in [6] we have

$$(9.16)\qquad \frac{1}{h}[u(2,0)-u(1,0)] = \frac{1}{h}[1-\frac{2}{\pi}-\frac{1}{4}] = \frac{b_1}{h}$$

$$(9.17)\qquad \frac{1}{h}[u(2,1)-u(1,1)] = \frac{1}{h}[-\frac{1}{4}+\frac{2}{\pi}-\frac{1}{4}] = \frac{b_2}{h}$$

$$(9.18)\qquad \frac{1}{h}[u(1,0)-u(0,0)] = \frac{1}{h}[\frac{1}{4}-0] = \frac{b_3}{h}$$

$$(9.19)\qquad \frac{1}{h}[u(1,1)-u(0,1)] = \frac{1}{h}[\frac{1}{\pi}-\frac{1}{4}] = \frac{b_4}{h} \;.$$

Hence

$$(9.20)\qquad h^2 \sum_{0\le r\le 2h}\sum \left|\frac{\triangle}{\triangle x}u(x,y)\right| = h^2\left\{\frac{2b_1+2b_2+4b_3+4b_4}{h}\right\} = bh \;.$$

By (9.13) and (9.20) we obtain for (9.14)

$$(9.21)\qquad h^2 \sum_{0\le r\le 1}\sum \left|\frac{\triangle}{\triangle x}u(x,y)\right| \le m_1+m_2h \;.$$

It remains to estimate

$$(9.22)\qquad h^2 \sum\sum \left|\frac{\triangle}{\triangle x}u(x,y)\right| = h^2 \sum_{\tilde{R}' \leq \tilde{r} \leq R}\sum \frac{1}{h}\left|\frac{\triangle}{\triangle x}u(\tilde{x},\tilde{y})\right|$$

We recall that in the above region

$$|u(\tilde{x}+1,\tilde{y})-u(\tilde{x},\tilde{y})| \leq \frac{1}{2\pi}\left|\log\sqrt{(\tilde{x}+1)^2+\tilde{y}^2}-\log\sqrt{\tilde{x}^2+\tilde{y}^2}\right|+\frac{M}{\tilde{x}^2+\tilde{y}^2}\quad\frac{M}{(\tilde{x}+1)^2+\tilde{y}^2}$$

hence

$$\left|\frac{\triangle}{\triangle x}u(x,y)\right| = \left|\frac{1}{h}\frac{\triangle}{\triangle\tilde{x}}u(\tilde{x},\tilde{y})\right| \leq \frac{1}{h}\left\{\frac{1}{2\pi}\left|\log\sqrt{(\tilde{x}+1)^2+\tilde{y}^2}-\log\sqrt{\tilde{x}^2+\tilde{y}^2}\right|+ + \frac{M}{\tilde{x}^2+\tilde{y}^2}+\frac{M}{(\tilde{x}+1)^2+\tilde{y}^2}\right\}$$

$$\left|\frac{\triangle}{\triangle x}u(x,y)\right| \leq \frac{1}{h}\left\{\frac{1}{4\pi}\log\left|\left(1+\frac{2\tilde{x}+1}{\tilde{x}^2+\tilde{y}^2}\right)\right|+\frac{M}{\tilde{x}^2+\tilde{y}^2}+\frac{M}{(\tilde{x}+1)^2+\tilde{y}^2}\right\}$$

$$\left|\frac{\triangle}{\triangle x}u(x,y)\right| \leq \frac{1}{h}\left\{\frac{1}{4\pi}\left|\frac{2\tilde{x}+1}{\tilde{r}^2}\right|+\frac{M}{\tilde{r}^2}+\frac{M}{\tilde{r}^2+2\tilde{x}+1}\right\}.$$

Transforming to polar coordinates, i.e., letting $\tilde{x} = \tilde{r}\cos\alpha$, we obtain

$$\left|\frac{\triangle}{\triangle x}u(x,y)\right| \leq \frac{1}{h}\left[\frac{1}{4\pi}\cdot\frac{2\tilde{r}\cos\alpha}{\tilde{r}^2}+\frac{4\pi M+1}{4\pi\tilde{r}^2}+\frac{M}{\tilde{r}^2+2\tilde{r}\cos\alpha+1}\right]$$

$$\leq \frac{1}{h}\left[\frac{1}{4\pi}\frac{2}{\tilde{r}}+\frac{4\pi M+1}{4\pi\tilde{r}^2}+\frac{M}{(\tilde{r}-1)^2}\right]$$

returning to the original plane, i.e., letting $r = \tilde{r}h$ we obtain

$$\left|\frac{\triangle}{\triangle x}u(x,y)\right| \leq \frac{1}{h}\left[\frac{1}{2\pi}\frac{h}{r}+\frac{\tilde{M}h^2}{r^2}+\frac{Mh^2}{(r-h)^2}\right].$$

Hence for $1 \leq r \leq R$ we have

$$\left|\frac{\triangle}{\triangle x}u(x,y)\right| \leq \frac{1}{2\pi}\cdot\frac{1}{r}+\frac{\tilde{M}h}{r^2}+\frac{Mh}{(r-h)^2}$$

and

$$h^2 \sum\sum \left|\frac{\Delta}{\Delta x}u(x,y)\right| \le h^2 \sum_{1\le r\le R}\sum \frac{1}{2\pi}\cdot\frac{1}{r}+h^2 \sum_{1\le r\le R}\sum \frac{\tilde{M}h}{r^2} + \tag{9.23}$$

$$+ h^2 \sum_{1\le r\le R}\sum \frac{Mh}{(r-h)^2}$$

but

$$\lim_{h\to 0} h^2 \sum_{1\le r\le R}\sum \frac{1}{2\pi}\frac{1}{r} = \frac{1}{2\pi}\int_0^{2\pi} d\theta\int_1^R \frac{1}{r}r\,dr = (R-1)a_1$$

$$\lim_{h\to 0} h^2 \sum_{1\le r\le R}\sum \frac{\tilde{M}h}{r^2} = \tilde{M}h\int_0^{2\pi} d\theta\int_1^R \frac{r\,dr}{r^2} = 2\pi h\tilde{M}\log(R-1) = a_2h$$

$$\lim_{h\to 0} h^2 \sum_{1\le r\le R}\sum \frac{Mh}{(r-h)^2} = Mh\int_0^{2\pi} d\theta\int_1^R \frac{r\,dr}{(r-h)^2} = 2\pi Mh\left[\log(r-h)-\frac{h}{r-h}\right]_1^R$$

$$= 2\pi Mh\log\frac{R-h}{1-h}+h\left[\frac{1}{1-h}-\frac{1}{r-h}\right] \approx a_3h\ .$$

Combining all above bounds we obtain

$$h^2 \sum\sum \left|\frac{\Delta}{\Delta x}u(x,y)\right| \le c_1+c_2h\ . \tag{9.24}$$

Since $u(\tilde{x},\tilde{y}) = u(\tilde{y},\tilde{x})$, in a similar manner as above we would obtain the following bound for $\frac{\Delta}{\Delta y}u(x,y)$

$$h^2 \sum\sum \left|\frac{\Delta}{\Delta y}u(x,y)\right| \le c_1+c_2h\ . \tag{9.25}$$

Using (9.24) and (9.25) as well as the fact that $|g(P)| \le \le c_o\delta$ (derived at the end of the previous chapter) we obtain from (7.10) and (7.11) the following bounds in $G_h$ for $\frac{\Delta}{\Delta x}e_1(P)$ and $\frac{\Delta}{\Delta y}e_1(P)$, respectively

$$\left|\frac{\Delta}{\Delta x}e_1(P)\right| \le \tilde{c}_o\delta+O(h^3) \tag{9.26}$$

$$\left|\frac{\Delta}{\Delta y}e_1(P)\right| \le \tilde{c}_o\delta+O(h^3)\ . \tag{9.27}$$

## 10. A Bound for the Homogeneous Case

In order to finish all the preparations for finding the bounds for $\frac{\Delta}{\Delta x}e$ and $\frac{\Delta}{\Delta y}e$ from (7.3) we still must find a bound for the solution of the difference equation $\Delta_h e_2 = 0$ and its difference quotients. We proceed as follows. By (7.5) we have

$$\Delta_h e_2 = 0 \quad \text{in } G_h$$
$$e_2\big|_B = -e_1(P) \text{ on the boundary .}$$

Hence, from the above and (8.20), we obtain

(10.1) $$|e_2|_B \leq |e_1(P)| \leq c_o^*\delta|\log h| + O(h^2) \ .$$

But $\Delta_h e_2(P) = 0$ satisfies the maximum principle, i.e., $e_2(P)$ can take on its maximum value only on the boundary, hence

(10.2) $$|e_2(P)| \leq c_o^*\delta|\log h|$$

at all interior points of $G_h$.

Next we shall prove the following theorem:

<u>Theorem</u>. Let G be a square domain, i.e., $|x| \leq b$; $|y| \leq b$; $G'$ -- its subdomain and $u(P)$ the set of all lattice functions that satisfy the difference equations $\Delta_h u(P) = 0$ in G and are uniformly bounded, i.e., $|u(P)| \leq A$ in G. Then there exists a constant $A'$ such that

$$\left|\frac{\Delta}{\Delta x}u\right| < A' \quad \text{and} \quad \left|\frac{\Delta}{\Delta y}u\right| < A' \quad \text{in} \quad G' \ .$$

At this point we would like to note that in this, and only this, chapter we shall denote the difference quotients of a function u by $u_x(x,y)$, $u_{\bar{x}}(x,y)$, $u_y(x,y)$, and $u_{\bar{y}}(x,y)$, i.e.,

(10.3) $$u_x(x,y) = \frac{u(x+h,y)-u(x,y)}{h} \ ; \ u_{\bar{x}}(x,y) = \frac{u(x,y)-u(x-h,y)}{h}$$

(10.4) $u_y(x,y) = \dfrac{u(x,y+h)-u(x,y)}{h}$ ; $u_{\bar{y}}(x,y) = \dfrac{u(x,y)-u(x,y-h)}{h}$

whereas in the previous chapters the above was denoted by $\dfrac{\Delta}{\Delta x}u$ and $\dfrac{\Delta}{\Delta y}u$, respectively.

Similarly we define the second differences of a function u by:

$$u_{x\bar{x}} = \frac{u(x+h,y)+u(x-h,y)-2u(x,y)}{h^2} \tag{10.5}$$

$$u_{y\bar{y}} = \frac{u(x,y+h)+u(x,y-h)-2u(x,y)}{h^2} \quad . \tag{10.6}$$

Hence the operator $\Delta_h$ defined by (1.3) can be written as:

$$\Delta_h u = u_{x\bar{x}}+u_{y\bar{y}} \quad . \tag{10.7}$$

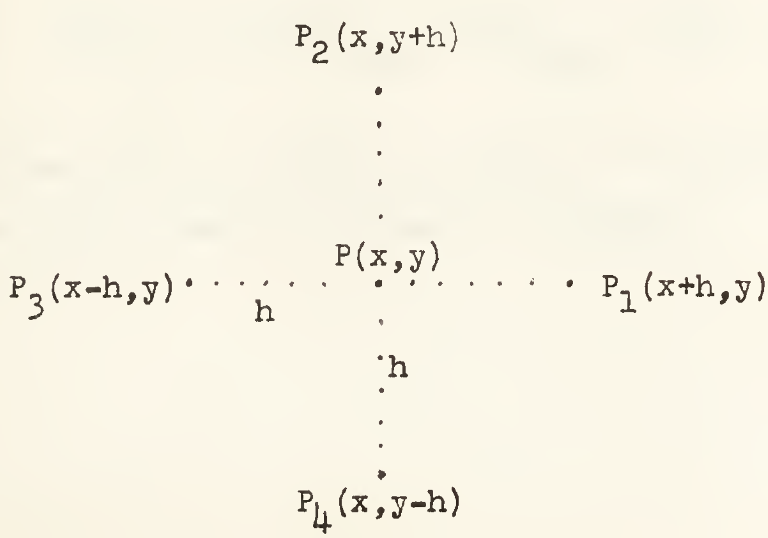
To prove the above theorem we shall employ the following auxiliary function

$$z(P) = u_x^2\Phi+C[u^2(P)+u^2(P_1)+u^2(P_2)+u^2(P_3)+u^2(P_4)] \tag{10.8}$$

where

$$\Phi = (x^2-b^2)^2(y^2-b^2)^2 \ ; \tag{10.9}$$

C -- a positive constant to be determined later and $P_1$, $P_2$, $P_3$ and $P_4$ are explained by the diagram below.



Now we shall show that the function z(P) satisfies the inequality $\Delta_h z(P) \geq 0$. Applying the operator $\Delta_h$ as defined by (10.7) on both sides of (10.8) we obtain:

$$(10.10) \quad \Delta_h z = \Delta_h(u_x^2 \Phi) + C\Delta_h[u^2(P) + u^2(P_1) + u^2(P_2) + u^2(P_3) + u^2(P_4)]$$

Let us find first

$$(10.11) \quad \Delta_h(u_x^2 \Phi) = (u_x^2 \Phi)_{x\bar{x}} + (u_x^2 \Phi)_{y\bar{y}} \quad .$$

To compute the above expression we make use of the following formula which can be easily verified:

$$(10.12) \quad (fg)_{x\bar{x}} = fg_{x\bar{x}} + f_x g_x + f_{\bar{x}} g_{\bar{x}} + f_{x\bar{x}} g \quad .$$

Hence

$$(10.13) \quad (u_x^2 \Phi)_{x\bar{x}} = u_x^2 \Phi_{x\bar{x}} + (u_x^2)_x \Phi_x + (u_x^2)_{\bar{x}} \Phi_{\bar{x}} + (u_x^2)_{x\bar{x}} \Phi$$

Using (10.12) once more we obtain for $(u_x^2)_{x\bar{x}}$:

$$(10.14) \quad (u_x^2)_{x\bar{x}} = (u_x \cdot u_x)_{x\bar{x}} = 2u_x u_{xx\bar{x}} + u_{xx}^2 + u_{x\bar{x}}^2 \quad .$$

To find $(u_x^2)_x$ and $(u_x^2)_{\bar{x}}$ we make use of the following two formulas, which can also be easily verified:

$$(fg)_x = f(P_1)\, g_x + f_x\, g \tag{10.15}$$

$$(fg)_{\bar{x}} = f(P_3)\, g_{\bar{x}} + f_{\bar{x}}\, g \quad . \tag{10.16}$$

Then

$$(u_x^2)_x = (u_x \cdot u_x)_x = u_x\, u_{xx} + u_{xx} u_x\, (P_1) \tag{10.17}$$

and

$$(u_x^2)_{\bar{x}} = (u_x \cdot u_x)_{\bar{x}} = u_x\, u_{x\bar{x}} + u_{x\bar{x}}\, u_x\, (P_3) \tag{10.18}$$

Using (10.14), (10.17) and (10.18) we obtain from (10.13)

$$\begin{aligned}(u_x^2\, \Phi)_{x\bar{x}} = u_x^2\, \Phi_{x\bar{x}} &+ [u_x\, u_{xx} + u_{xx}\, u_x\, (P_1)]\, \Phi_x + \\ &+ [u_x\, u_{x\bar{x}} + u_{x\bar{x}}\, u_x\, (P_3)]\, \Phi_{\bar{x}} + \\ &+ [2u_x\, u_{xx\bar{x}} + u_{xx}^2 + u_{x\bar{x}}^2]\, \Phi\end{aligned} \tag{10.19}$$

In a similar manner using analogous formulas for the differences in y we would obtain for $(u_x^2\, \Phi)_{y\bar{y}}$ the following expression:

$$\begin{aligned}(u_x^2\, \Phi)_{y\bar{y}} = u_x^2\, \Phi_{y\bar{y}} &+ [u_x\, u_{xy} + u_{xy}\, u_x\, (P_2)]\, \Phi_y + \\ &+ [u_x\, u_{x\bar{y}} + u_{x\bar{y}}\, u_x\, (P_4)]\, \Phi_{\bar{y}} + \\ &+ [2u_x\, u_{xy\bar{y}} + u_{xy}^2 + u_{x\bar{y}}^2]\, \Phi \quad .\end{aligned} \tag{10.20}$$

Combining (10.19) and (10.20) we obtain from (10.1) the following expression for $\Delta_h(u_x^2\,\Phi)$:

$$
\begin{aligned}
\Delta_h(u_x^2\,\Phi) = u_x^2[\Phi_{x\bar{x}} + \Phi_{y\bar{y}}] &+ [u_x\,u_{xx} + u_{xx}\,u_x\,(P_1)]\,\Phi_x + \\
&+ [u_x\,u_{x\bar{x}} + u_{x\bar{x}}\,u_x\,(P_3)]\,\Phi_{\bar{x}} + \\
&+ [u_x\,u_{xy} + u_{xy}\,u_x\,(P_2)]\,\Phi_y + \\
&+ [u_x\,u_{x\bar{y}} + u_{x\bar{y}}\,u_x\,(P_4)]\,\Phi_{\bar{y}} + \\
&+ [2u_x(u_{xx\bar{x}} + u_{xy\bar{y}}) + u_{xx}^2 + u_{x\bar{x}}^2 + u_{xy}^2 + u_{x\bar{y}}^2]\,\Phi
\end{aligned}
\tag{10.21}
$$

Remembering that one of our assumptions has been that $\Delta_h u = 0$, i.e.,

$$u_{x\bar{x}} + u_{y\bar{y}} = 0 \tag{10.22}$$

and taking the difference quotient with respect to x of the above expression we obtain

$$u_{x\bar{x}x} + u_{y\bar{y}x} = 0 \tag{10.23}$$

It is easily seen that

$$u_{x\bar{x}x} = u_{xx\bar{x}} \quad \text{and} \quad u_{y\bar{y}x} = u_{xy\bar{y}}$$

therefore

$$u_{xx\bar{x}} + u_{xy\bar{y}} = 0$$

and (10.21) becomes:

$$
\begin{aligned}
\Delta_h\,[u_x^2\,\Phi] = u_x^2\,[\Phi_{x\bar{x}} + \Phi_{y\bar{y}}] &+ [u_x\,u_{xx} + u_{xx}\,u_x\,(P_1)]\,\Phi_x + \\
&+ [u_x\,u_{x\bar{x}} + u_{x\bar{x}}\,u_x\,(P_3)]\,\Phi_{\bar{x}} + \\
&+ [u_x\,u_{xy} + u_{xy}\,u_x\,(P_2)]\,\Phi_y + \\
&+ [u_x\,u_{x\bar{y}} + u_{x\bar{y}}\,u_x\,(P_4)]\,\Phi_{\bar{y}} + \\
&+ (u_{xx}^2 + u_{x\bar{x}}^2 + u_{xy}^2 + u_{x\bar{y}}^2)\,\Phi
\end{aligned}
\tag{10.24}
$$

To complete the computation of $\Delta_h z$ we still have to find $\Delta_h u^2$, $\Delta_h u^2(P_1)$, $\Delta_h u^2(P_2)$, $\Delta_h u^2(P_3)$ and $\Delta_h u^2(P_4)$.

$$\Delta_h u^2 = (u^2)_{x\bar{x}}+(u^2)_{y\bar{y}} = (u\cdot u)_{x\bar{x}}+(u\cdot u)_{y\bar{y}} \ . \tag{10.25}$$

Applying (10.12) to (10.25) we obtain

$$\Delta_h u^2 = 2u(u_{x\bar{x}}+u_{y\bar{y}})+u_x^2+u_{\bar{x}}^2+u_y^2+u_{\bar{y}}^2 \ . \tag{10.26}$$

Considering (10.22), (10.26) becomes

$$\Delta_h u^2 = u_x^2+u_{\bar{x}}^2+u_y^2+u_{\bar{y}}^2 \ . \tag{10.27}$$

In a similar manner as above we shall get the following analogous expressions for $\Delta_h u^2(P_1)$, $\Delta_h u^2(P_2)$, $\Delta_h u^2(P_3)$ and $\Delta_h u^2(P_4)$, respectively.

$$\Delta_h u^2(P_1) = u_x^2(P_1)+u_{\bar{x}}^2(P_1)+u_y^2(P_1)+u_{\bar{y}}^2(P_1) \tag{10.28a}$$

$$\Delta_h u^2(P_2) = u_x^2(P_2)+u_{\bar{x}}^2(P_2)+u_y^2(P_2)+u_{\bar{y}}^2(P_2) \ . \tag{10.28b}$$

$$\Delta_h u^2(P_3) = u_x^2(P_3)+u_{\bar{x}}^2(P_3)+u_y^2(P_3)+u_{\bar{y}}^2(P_3) \tag{10.28c}$$

$$\Delta_h u^2(P_4) = u_x^2(P_4)+u_{\bar{x}}^2(P_4)+u_y^2(P_4)+u_{\bar{y}}^2(P_4) \ . \tag{10.28d}$$

To obtain the upper bounds for some terms in (10.24) we have to examine the function

$$\Phi = (x^2-b^2)^2(y^2-b^2)^2 \ .$$

Obviously $\Phi$ is a continuous function for all $x,y$ in the closed bounded domain $G$: $|x| \leq b$, $|y| \leq b$ and is twice differentiable. Let us differ ntiate $\Phi$ with respect to $x$

$$\frac{\partial \Phi}{\partial x} = 4x(x^2-b^2)(y^2-b^2)^2 = 4x(y^2-b^2)\sqrt{\Phi} \ . \tag{10.30}$$

From the above we see that $4x(y^2-b^2)$ is continuous in the

closed bounded domain G; hence it assumes its greatest value at least once and there is a constant $\theta_1$ such that in G, $\max_{x,y\varepsilon G} 4x \cdot (y^2-b^2) = \theta_1$ and from (10.30)

$$|\frac{\partial\Phi}{\partial x}| = |4x(y^2-b^2)|\sqrt{\Phi} \leq \theta_1\sqrt{\Phi}\,. \tag{10.31}$$

As we have seen $\Phi$ is a continuous function in the closed bounded domain G and $\partial\Phi/\partial x$ exists; hence, applying the mean value theorem to $\Phi$, we will be able to obtain a bound for the difference quotient of $\Phi$

$$\Phi_x = \frac{\Phi(x+h,y)-\Phi(x,y)}{h} = \frac{\partial\Phi(x+\vartheta h,y)}{\partial x}\,. \tag{10.32}$$

Considering (10.31) we obtain

$$|\Phi_x| < \theta_1\sqrt{\Phi}\,. \tag{10.33}$$

In a similar manner we would obtain

$$|\Phi_{\bar{x}}| < \theta_1\sqrt{\Phi}\,. \tag{10.34}$$

Using reasoning similar to the above, only with respect to y we shall obtain the following bounds for the difference quotients of $\Phi$ in the y-direction

$$|\Phi_y| < \theta_2\sqrt{\Phi} \quad \text{and} \quad |\Phi_{\bar{y}}| < \theta_2\sqrt{\Phi} \tag{10.35}$$

where $\theta_2 = \max 4y(x^2-b^2)$.

Using (10.32) we shall show that $\Phi_{x\bar{x}}$ is bounded. By (10.32)

$$\Phi_{x\bar{x}} = \frac{\dfrac{\partial\Phi(x+\vartheta h,y)}{\partial x} - \dfrac{\partial\Phi(x+\vartheta h-h,y)}{\partial x}}{h} = \frac{\partial^2\Phi(\xi_1,y)}{\partial x^2}\,. \tag{10.36}$$

Differentiating (10.30) with respect to x we get

$$\frac{\partial^2\Phi}{\partial x^2} = 4(y^2-b^2)\sqrt{\Phi}+8x^2(y^2-b^2)^2\ ; \tag{10.37}$$

hence $\partial^2\Phi/\partial x^2$ is a continuous function in the closed bounded do-

main G; $|x| \le b$; $|y| \le b$; therefore it achieves its greatest value at least once, i.e.

$$(10.38) \qquad \left|\frac{\partial^2\Phi(\xi,y)}{\partial x^2}\right| \le \max_{x,y\varepsilon G} \frac{\partial^2\Phi(x,y)}{\partial x^2} = \theta_3 .$$

Using the above we obtain from (10.36)

$$(10.39) \qquad |\Phi_{x\bar{x}}| < \theta_3 .$$

In an analogous manner we obtain the following estimate for $\Phi_{y\bar{y}}$

$$(10.40) \qquad |\Phi_{y\bar{y}}| < \theta_4$$

where

$$\theta_4 = \max_{x,y\varepsilon G} \frac{\partial^2\Phi(x,y)}{\partial y^2} .$$

Combining (10.39) and (10.40) we obtain

$$(10.41) \qquad \begin{gathered} |\Delta_h\Phi| = |\Phi_{x\bar{x}}|+|\Phi_{y\bar{y}}| \le \theta_3+\theta_4 = \theta_5 \\ |\Delta_h\Phi| < \theta_5 . \end{gathered}$$

Choosing $\theta$ such that $\theta = \max(\theta_1,\theta_2,\theta_3,\theta_5)$ we shall have the following bounds for the various difference quotients of $\Phi$,

$$(10.42) \qquad \begin{gathered} |\Delta_h\Phi| < \theta , \; |\Phi_x| < \theta\sqrt{\Phi} , \; |\Phi_{\bar{x}}| < \theta\sqrt{\Phi} , \\ |\Phi_y| < \theta\sqrt{\Phi} , \; |\Phi_{\bar{y}}| < \theta\sqrt{\Phi} . \end{gathered}$$

Using the above bounds and the fact that for any two numbers $\alpha,\beta$

$$(10.43) \qquad \alpha\beta \le \frac{\alpha^2+\beta^2}{2} \le \alpha^2+\beta^2$$

we obtain for any $\varepsilon > 0$

$$(10.44) \quad |\Phi_x u_x u_{xx}| = \left|\frac{u_x}{\varepsilon}\cdot\varepsilon\Phi_x\cdot u_{xx}\right| \le \frac{u_x^2}{\varepsilon^2}+\varepsilon^2\Phi_x^2 u_{xx}^2 \le \frac{u_x^2}{\varepsilon^2}+\varepsilon^2\theta^2\Phi u_{xx}^2 .$$

(10.45) $$|\Phi_x u_x u_{x\bar{x}}| \le \frac{u_x^2}{\varepsilon^2}+\varepsilon^2\Theta^2\Phi u_{x\bar{x}}^2$$

(10.46) $$|\Phi_y u_x u_{xy}| \le \frac{u_x^2}{\varepsilon^2}+\varepsilon^2\Theta^2\Phi u_{xy}^2$$

(10.47) $$|\Phi_{\bar{y}} u_x u_{x\bar{y}}| \le \frac{u_x^2}{\varepsilon^2}+\varepsilon^2\Theta^2\Phi u_{x\bar{y}}^2$$

(10.48) $$|\Phi_x u_x(P_1) u_{xx}| \le \frac{u_x^2(P_1)}{\varepsilon^2}+\varepsilon^2\Theta^2\Phi u_{xx}^2$$

(10.49) $$|\Phi_x u_x(P_3) u_{x\bar{x}}| \le \frac{u_x^2(P_3)}{\varepsilon^2}+\varepsilon^2\Theta^2\Phi u_{x\bar{x}}^2$$

(10.50) $$|\Phi_y u_x(P_2) u_{xy}| \le \frac{u_x^2(P_2)}{\varepsilon^2}+\varepsilon^2\Theta^2\Phi u_{xy}^2$$

(10.51) $$|\Phi_y u_x(P_4) u_{x\bar{y}}| \le \frac{u_x^2(P_4)}{\varepsilon^2}+\varepsilon^2\Theta^2\Phi u_{x\bar{y}}^2 .$$

Considering the above bounds we conclude that (10.24) reduces to

(10.52) $$\begin{aligned}\Delta_h(u_x^2\Phi) \ge (1-2\Theta^2\varepsilon^2)\Phi(u_{xx}^2+u_{x\bar{x}}^2+u_{xy}^2+u_{x\bar{y}}^2)-(\frac{4}{\varepsilon^2}+\Theta)u_x^2-\\ -\frac{1}{\varepsilon^2}[u_x^2(P_1)+u_x^2(P_2)+u_x^2(P_3)+u_x^2(P_4)] .\end{aligned}$$

If we add to both sides of the above inequality the following positive expression

$$C\Delta_h[u^2+u^2(P_1)+u^2(P_2)+u^2(P_3)+u^2(P_4)]$$

and substitute on the right side the computed values for $\Delta_h u^2$, $\Delta_h u^2(P_1)$, $\Delta_h u^2(P_2)$, $\Delta_h u^2(P_3)$ and $\Delta_h u^2(P_4)$ from (10.28a) to (10.28d), respectively, we obtain

$$
\begin{aligned}
&\Delta_h(u_x^2\Phi)+C\Delta_h[u^2+u^2(P_1)+u^2(P_2)+u^2(P_3)+u^2(P_4)] \geq \\
&\quad \geq (1-2\varepsilon^2\theta^2)\Phi(u_{xx}^2+u_{x\bar{x}}^2+u_{xy}^2+u_{x\bar{y}}^2)-(C-\frac{4}{\varepsilon^2}-\theta)u_x^2+(C-\frac{1}{\varepsilon^2})\cdot \\
&\quad \cdot [u_x^2(P_1)+u_x^2(P_2)+u_x^2(P_3)+u_x^2(P_4)]+C[u_{\bar{x}}^2+u_{\bar{x}}^2(P_1)+ \\
&\quad + u_{\bar{x}}^2(P_2)+u_{\bar{x}}^2(P_3)+u_{\bar{x}}^2(P_4)]+C[u_y^2+u_y^2(P_1)+u_y^2(P_2)+ \\
&\quad + u_y^2(P_3)+u_y^2(P_4)]+C[u_{\bar{y}}^2+u_{\bar{y}}^2(P_1)+u_{\bar{y}}^2(P_2)+u_{\bar{y}}^2(P_3)+ \\
&\quad + u_{\bar{y}}^2(P_4)] \ .
\end{aligned} \tag{10.53}
$$

According to (10.10) we identify the left side of the above inequality as $\Delta_h z$, and if we select $\varepsilon$ and $C$ so that

$$\varepsilon^2\theta^2 \leq \frac{1}{2} \qquad \text{and} \qquad C \geq \frac{4}{\varepsilon^2}+\theta \tag{10.54}$$

which also implies that $C-1/\varepsilon^2 \geq 0$; we conclude from (10.53)

$$\Delta_h z \geq 0 \ . \tag{10.55}$$

Writing out the above operator explicitly we get

$$z(P) \leq \frac{1}{4}[z(P_1)+z(P_2)+z(P_3)+z(P_4)]$$

at all lattice points.

Consequently $z(P)$ can have no maximum at an interior point of any set, although it may have a minimum. Hence the maximum value of $z(P)$ must occur on the boundary. But by (10.9) $\Phi = 0$ on the boundary. Therefore in tho whole square $|x| \leq b$; $|y| \leq b$

$$0 \leq z(P) \leq z_B$$

where $z_B$ denotes the value of $z$ on the boundary. Considering (10.8) we obtain

$$0 \leq z(P) \leq 5CA^2 \ .$$

Since the second term of (10.8) is non-negative, we conclude that for $P \in G'$

$$(10.56) \qquad u_x^2(P) \leq \frac{5CA^2}{\Phi}$$

where for $\Phi$ we have to take its lower bound.

Let us denote by d the distance between G and G'; then from

$$\Phi = (x^2-b^2)^2(y^2-b^2)^2 = (x-b)^2(x+b)^2(y-b)^2(y+b)^2 ,$$

taking

$$|b-x| \geq d$$

$$|b+x| \geq d$$

as well as

$$|b-y| \geq d$$

$$|b+y| \geq d$$

we can majorize $\Phi$ by $d^8$. Since $d^8 > 0$ we can take the right side of (10.56) for $(A')^2$ and thus the following estimate for $u_x$ is established

$$(10.57) \qquad |u_x| \leq A' = \frac{\sqrt{5C}A}{d^4} .$$

In a similar manner we would obtain

$$(10.58) \qquad |u_y| \leq A' , \quad |u_{\bar{x}}| \leq A' \quad \text{and} \quad |u_{\bar{y}}| \leq A' .$$

Since $e_2(P)$ satisfies all the assumptions of the theorem proved above, with $A = c_o^* \delta |\log h|$ we conclude that in the subdomain G' we shall have

$$(10.59) \qquad |(e_2)_x| \leq A' = \frac{\sqrt{5C}A}{d^4} = \bar{c}_o \delta |\log h|$$

$$(10.60) \qquad |(e_2)_y| \leq \bar{c}_o \delta |\log h|$$

$$(10.61) \qquad |(e_2)_{\bar{x}}| \leq \bar{c}_o \delta |\log h|$$

$$(10.62) \qquad |(e_2)_{\bar{y}}| \leq \bar{c}_o \delta |\log h| .$$

## 11. Estimates for $(\Delta/\Delta x)e$ and $(\Delta/\Delta y)e$

Before using the subsidiary estimates of the previous chapters to obtain the bound for $\frac{\Delta}{\Delta x}e$ and $\frac{\Delta}{\Delta y}e$, we recall that

$$\frac{\Delta}{\Delta x}e = \frac{e(x+h,y)-e(x-h,y)}{2h} \tag{11.1}$$

$$\frac{\Delta}{\Delta y}e = \frac{e(x,y+h)-e(x,y-h)}{2h} \tag{11.2}$$

and

$$e = e_1+e_2 \ . \tag{11.3}$$

Therefore taking difference quotients of both sides, we get

$$\frac{\Delta}{\Delta x}e = \frac{\Delta}{\Delta x}e_1+\frac{\Delta}{\Delta x}e_2 \ ;$$

hence

$$\left|\frac{\Delta}{\Delta x}e\right| \leq \left|\frac{\Delta}{\Delta x}e_1\right|+\left|\frac{\Delta}{\Delta x}e_2\right| \tag{11.4}$$

and similarly

$$\left|\frac{\Delta}{\Delta y}e\right| \leq \left|\frac{\Delta}{\Delta y}e_1\right|+\left|\frac{\Delta}{\Delta y}e_2\right| \ . \tag{11.5}$$

However, in the estimates for $\frac{\Delta}{\Delta x}e_1$ and $\frac{\Delta}{\Delta x}e_2$ as well as for $\frac{\Delta}{\Delta y}e_1$ and $\frac{\Delta}{\Delta y}e_2$, given by (9.26), (9.27), (10.59) and (10.60) the forward differences have been taken, while, according to (11.1) and (11.2) centered differences are required. Therefore having at our disposal the estimates for backward differences as well, and noting that

$$\frac{\Delta}{\Delta x}e = \frac{1}{2}(e_x+e_{\bar{x}})$$

and similarly

$$\frac{\Delta}{\Delta y}e = \frac{1}{2}(e_y+e_{\bar{y}}) \ .$$

Finally we would obtain

(11.6) $$\left|\frac{\Delta}{\Delta x}e\right| \le C_o\delta|\log h|+O(h^2)$$

and

(11.7) $$\left|\frac{\Delta}{\Delta y}e\right| \le C_o\delta|\log h|+O(h^2) \;.$$

## 12. Reflection Principle

In previous chapters we found the estimates for $e$, $\frac{\Delta}{\Delta x}e$ and $\frac{\Delta}{\Delta y}e$ where $e$ satisfied the following equation

$$\Delta_h e = g(P) \quad \text{in the interior of } G_h\text{: } |x| \le 1,\ |y| \le 1$$
$$e = 0 \quad \text{on the boundary .}$$

However, our estimates for $\frac{\Delta}{\Delta x}e$ and $\frac{\Delta}{\Delta y}e$ do not hold sufficiently close to the boundary. To make them hold up to the boundary we shall proceed as follows.

Suppose we are given a function $e(x,y)$ which is the function $e$ described above, and let us define a function $E(x,y)$ in the domain $D$: $|x| \le 3$, $|y| \le 3$ such that

$$\Delta_h E = G(P) \quad \text{in } D$$
$$E = 0 \quad \text{on the boundary of } D$$

and

(12.1) $E(x,y) = e(x,y)$ in the interior of rectangle I:

$$-1 \le x \le 1\ ;\ -1 \le y \le 1$$

(12.2) $E(x^*,y^*) = E(-2-x,y) = -e(x,y)$ in the interior of a rectangle $I_a$: $-3 \le x^* \le -1\ ;\ 1 \le y^* \le 3$

(12.3) $E(x^*,y^*) = E(x,2-y) = -e(x,y)$ in the interior of rectangle $I_b$: $-1 \le x^* \le 1\ ;\ 1 \le y^* \le 3$

(12.4) $E(x^*,y^*) = E(2-x,y) = -e(x,y)$ in the interior of rectangle $I_c$: $1 \le x^* \le 3\ ;\ -1 \le y^* \le 1$

(12.5) $E(x^*,y^*) = E(x,-2-y) = -e(x,y)$ in the interior

rectangle $I_d$: $-1 \le x^* \le 1$ ; $-3 \le y^* \le -1$ .

But in $I_a$

(12.6) $$\Delta_h E(x,y) = -\Delta_h e(P) = -g(P) = G(P)$$

and similarly in $I_b$, $I_c$ and $I_d$ we see that $G(P) = -g(P)$; it is obvious that $G(P) = g(P)$ in I (see Fig. 2).

Let us take now a point B on the boundary of I and $I_a$ and compute the value of E at B (denoted by $E_B$) by considering the points numbered 1a, 2a, 3a, 4a, and B in $I_a$ and points numbered 1, 2, 3, 4 and B in I (see Fig. 3) and writing out $\Delta_h E = G(P)$ explicitly for those points we obtain:

(12.7) $$E_{1a}+E_{2a}+E_{3a}+E_B-4E_{4a} = G_{4a}$$

(12.8) $$E_1+E_2+E_3+E_B-4E_4 = G_4 .$$

But by definition:

(12.9) $$\begin{aligned} E_1 &= e_1 \\ E_2 &= e_2 \\ E_3 &= e_3 \\ E_4 &= e_4 \quad \text{and} \quad G_4 = g_4 \end{aligned}$$

as well as

(12.10) $$\begin{aligned} E_{1a} &= -e_1 \\ E_{2a} &= -e_2 \\ E_{3a} &= -e_3 \\ E_{4a} &= -e_4 \quad \text{and} \quad G_{4a} = -g_4 ; \end{aligned}$$

therefore (12.7) and (12.8) become

$$-e_1-e_2-e_3+E_B+4e_4 = g_4$$

$$e_1+e_2+e_3+E_B-4e_4 = -g_4 ;$$

adding the above equalities we get

$$\begin{aligned} 2E_B &= 0 \\ E_B &= E(-1,y) = 0 \; . \end{aligned} \tag{12.11}$$

In a similar manner taking corresponding points in I and $I_b$, I and $I_c$, I and $I_d$ we would obtain

$$E(-1,y) = E(+1,y) = E(x,1) = E(x,-1) = 0 \; . \tag{12.12}$$

Now we are able to compute the value of G on the boundary of $|x| \leq 1$, $|y| \leq 1$. Using points numbered 4 and 4a in I and $I_a$ respectively and three boundary points and applying $\triangle_h E = G(P)$ to them we obtain

$$E_{4a}+E_4+E_B+E_B-4E_B = G_B \; , \tag{12.13}$$

but by definition

$$E_{4a} = -e_4 \; ; \; E_4 = e_4$$

and from (12.11)

$$E_B = 0 \; ;$$

hence (12.13) becomes

$$\begin{aligned} -e_4+e_4 &= G_B \\ G_B &= G(-1,y) = 0 \; . \end{aligned} \tag{12.14}$$

Similarly taking points in I and $I_b$, I and $I_c$, I and $I_d$ we would obtain

$$G(-1,y) = G(1,y) = G(x,+1) = G(x,-1) = 0 \; . \tag{12.15}$$

It is clear that what we did above was obtained by reflecting the rectangle I about lines $y = \pm 1$ and $x = \pm 1$. Repeating this process once more, i.e., reflecting $I_a$ about $y = \pm 1$, and $I_c$ about $y = \pm 1$, or $I_b$ about $x = \pm 1$ and $I_d$ about $x = \pm 1$ in a similar manner as before we would obtain the following rectangles which will be denoted by $II_a$ and $II_a^*$ and $II_c$ and $II_c^*$ or $II_b$ and $II_b^*$ and $II_d$ and $II_d^*$, respectively (see Fig. 2).

Using the same methods as above it is easily verified that in the interior of $II_a$: $-3 \leq x^{**} \leq -1$; $1 \leq y^{**} \leq 3$, $E(x^{**},y^{**}) = -E(x^*,y^*) = e(x,y)$, and $G(P^{**}) = -G(P^*) = g(P)$ and that $E(x^{**},1) = E(-1,y^{**}) = 0$ as well as $G(x^{**},1) = G(-1,y^{**}) = 0$; in the interior of $II_a^*$: $-3 \leq x^{**} \leq -1$, $-3 \leq y^{**} \leq -1$, $E(x^{**},y^{**}) = -E(x^*,y^*) = e(x,y)$ and $G(P^{**}) = -G(P^*) = g(P)$ and that $E(x^{**},-1) = E(-1,y^{**}) = 0$ as well as $G(x^{**},-1) = G(-1,y^{**}) = 0$; in the interior of $I_b$: $1 \leq x^{**} \leq 3$; $1 \leq y^{**} \leq 3$, $E(x^{**},y^{**}) = -E(x^*,y^*) = e(x,y)$ and $G(P^{**}) = -G(P^*) = g(P)$ and that $E(x^{**},1) = E(1,y^{**}) = 0$ as well as $G(x^{**},1) = G(1,y^{**}) = 0$; in the interior of $II_b^*$: $1 \leq x^{**} \leq 3$, $-3 \leq y^{**} \leq -1$, $E(x^{**},y^{**}) = -E(x^*,y^*) = e(x,y)$ and $G(P^{**}) = -G(P^*) = g(P)$ and that $E(x^{**},-1) = E(1,y^{**}) = 0$ as well as $G(x^{**},-1) = G(1,y^{**}) = 0$.

Therefore we conclude that the function $E(x,y)$ defined in the domain D: $|x| \leq 3$, $|y| \leq 3$ satisfies $\Delta_h E = G(P)$ in D and $E = 0$ on the boundary of D and that E as well as G is zero on the lines $x = \pm 1$, $y = \pm 1$ and otherwise $E(P^*)$ is either $+e(P)$ or $-e(P)$, as well as $G(P^*)$ is either $+g(P)$ or $-g(P)$. Hence in the interior of D

$$|E(P^*)| = |e(P)| \leq \delta$$

and

$$|G(P^*)| = |g(P)| \leq c_o\delta \ .$$

Therefore we are able to apply all the theory developed in chapters 7-11 to the function E and conclude that in the subdomain D': $|x| \leq 2$, $|y| \leq 2$, the following estimates hold:

$$\left|\frac{\Delta}{\Delta x}E\right| \leq C_o\delta|\log h| \tag{12.16}$$

(12.17) $$\left|\frac{\Delta}{\Delta y}E\right| \leq C_o\delta|\log h| .$$

But $E(x,y) = e(x,y)$ in $G_h$: $|x| \leq 1$, $|y| \leq 1$; hence the following estimates for $\frac{\Delta}{\Delta x}e$ and $\frac{\Delta}{\Delta y}e$ hold up to the boundary of $G_h$:

$$\left|\frac{\Delta}{\Delta x}e\right| \leq C_o\delta|\log h|$$

$$\left|\frac{\Delta}{\Delta y}e\right| \leq C_o\delta|\log h| .$$

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On the convergence of the numerical solution for a certain partial differential equation of third order.